SQIsign

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SQIsign: What?

https://sqisign.org

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- A new and very hot post-quantum signature scheme.
- In **round 2** of the NISTPQC signature on-ramp!

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Big picture

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$\rightsquigarrow$ *Cryptography!*

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- Essential tool for both constructions and attacks.

Constructively, *partially* known endomorphism rings are useful.
$\rightsquigarrow$ **Oriented curves** and **the isogeny class-group action**. 🏝️

(See my autumn-school lecture yesterday.)

The Deuring correspondence

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In fact, the image in $B_{p,\infty}$ of a $\mathbb{Z}$-basis of $\mathrm{End}(E)$ is given by

$$\{1, \quad \iota, \quad (\iota+\pi)/2, \quad (1+\iota\pi)/2\}\,.$$

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In fact, a $\mathbb{Z}$-basis of $\mathrm{End}(E')$ is given by

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Deuring correspondence: Example #3

For the sake of an example, let $p = 7799999 \equiv 11 \pmod{12}$.

Then $E\colon y^2 = x^3 + x$ and $E'\colon y^2 = x^3 + 1$ are both supersingular with endomorphism rings as shown before.

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Moreover, the lattice

$$\mathbb{Z}\,4947 \oplus \mathbb{Z}\,4947\iota \oplus \mathbb{Z}\,\frac{598 + 4947\iota + \pi}{2} \oplus \mathbb{Z}\,\frac{4947 + 598\iota + \iota\pi}{2}$$

inside $\mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponds to an isogeny $E \to E'$.

(I haven't yet said *how*.)

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<u>Core of the connection:</u> The Deuring correspondence!

$\Leftarrow$: Isogenies "transport" knowledge of endomorphism rings.

$\Rightarrow$: Finding powersmooth "connecting ideals" is easy (🪄);
converting them to isogenies is easy.

⚠️ About **4** math-heavy slides ahead.
It will become less technical afterwards! :)

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Indeed, this means

$$B_{p,\infty} = \mathbb{Q} \oplus \mathbb{Q}\mathbf{i} \oplus \mathbb{Q}\mathbf{j} \oplus \mathbb{Q}\mathbf{ij}$$

with multiplication defined by $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = -p$, $\mathbf{ji} = -\mathbf{ij}$.
(This $B_{p,\infty}$ is the “quaternion algebra over $\mathbb{Q}$ ramified at p and ∞”.)

The main theorem

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Theorem. The (contravariant) functor

$$E \longmapsto \mathrm{Hom}(E, E_0)$$

defines an equivalence of categories between

- supersingular elliptic curves with isogenies; and
- invertible left $\mathcal{O}_0$-modules

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⚠ There is no equivalence between elliptic curves/$\sim$ and endomorphism rings/$\sim$. (The map $\{E\}/\sim \to \{\mathcal{O}\}/\sim$ is not injective.)

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Important consequence: The isogeny $\varphi_I\colon E_0 \to E$ defined by a left $\mathcal{O}_0$-ideal I has kernel $\bigcap_{\alpha\in I} \ker\alpha \le E_0$.

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$\rightsquigarrow$ Explicit **ideal-to-isogeny** conversion, provided all the points of $\mathrm{norm}(I)$-torsion are accessible (defined over small field extensions):

1. Write $I = \mathcal{O}_0 N + \mathcal{O}_0\alpha$ where $N = \mathrm{norm}(I) \in \mathbb{Z}$ and $\alpha \in \mathcal{O}_0$.
2. Compute the isogeny with kernel $E[I] = \ker(\alpha|_{E_0[N]})$.

All the $\mathrm{End}(E) \hookrightarrow B_{p,\infty}$

From any isogeny $\varphi\colon E_0 \to E$, we obtain (abstractly) an embedding of the endomorphism ring

$$\begin{aligned} \mathrm{End}(E) &\longhookrightarrow B_{p,\infty} = \mathrm{End}(E_0) \otimes_{\mathbb{Z}} \mathbb{Q}\,, \\ \alpha &\longmapsto \widehat{\varphi}\alpha\varphi/\deg(\varphi)\,. \end{aligned}$$

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<u>"Transporting" endomorphism knowledge:</u>

Knowing $\mathrm{End}(E_0)$ and an isogeny $\varphi\colon E_0 \to E$ reveals $\mathrm{End}(E)$.

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Concretely: Under the embedding above, we have

$$\mathrm{End}(E) = \mathcal{O}_R(I_\varphi) = \{\alpha \in B_{p,\infty} : I_\varphi\alpha \subseteq I_\varphi\}\,,$$

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($\rightsquigarrow$ Open problem: Constructing supersingular E with unknown $\mathrm{End}(E)$.)

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$\rightsquigarrow$ Knowledge of $\mathrm{End}(E_A)$ is a *trapdoor* for finding $E_A \to E_2$.

SQIsign: main algorithms

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- Identification scheme based on isogenies:

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- Easy response: $E_A \to E_0 \to E_1 \to E_2$. *Obviously broken.*
- **SQIsign**'s solution: Construct new path $E_A \to E_2$ (using *secret*).

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- From $\mathrm{End}(E), \mathrm{End}(E')$, can randomize within $\mathrm{Hom}(E, E')$.

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> *"If you have KLPT implemented very nicely as a black box, then anyone can implement SQIsign."*
>
> — Yan Bo Ti

SQIsigning

Parameters:

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- Degrees for all (to be) involved isogenies, tons of precomputed constants, etc.

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Key generation:

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- Sample a random *secret* isogeny $\varphi\colon E_0 \to E_A$ together with its associated $\mathrm{End}(E_0)$-ideal I_φ.

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- The public key is just the codomain E_A.

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- Return the signature (E_1, σ).

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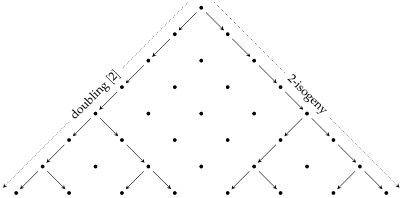
In SQIsign, the degrees are chosen so that $\deg(\sigma) = 2^n$.
$\rightsquigarrow$ very efficient isogeny chains in time $O(n \log n)$ using "optimal strategies".



Security

Required properties

For SQIsign to be secure, we need two main properties:

- Soundness: Ability to sign proves knowledge of a secret.
- Zero-knowledge: Signatures do not leak anything secret.

Soundness

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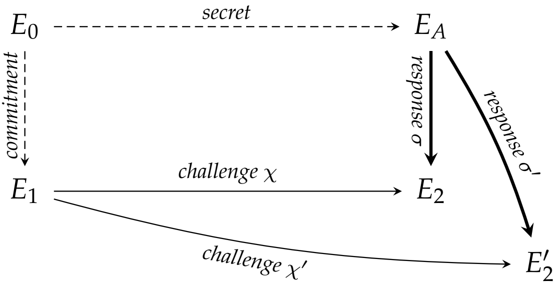

Soundness

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$$
\begin{tikzcd}
E_0 \arrow[rr, dashed, "secret"] \arrow[d, dashed, "commitment"'] & & E_A \arrow[d, "\text{response } \sigma"'] \arrow[dr, bend left, "\text{response } \sigma'"] & \\
E_1 \arrow[rr, "\text{challenge } \chi"] \arrow[rrr, bend right, "\text{challenge } \chi'"'] & & E_2 & E_2'
\end{tikzcd}
$$

☹ We cannot directly extract the secret $\varphi\colon E_0 \to E_A$, but we *can* extract an endomorphism in $\mathrm{End}(E_A)\backslash\mathbb{Z}$:

$$E_A \xrightarrow{\sigma'} E_2' \xrightarrow{\widehat{\chi}'} E_1 \xrightarrow{\chi} E_2 \xrightarrow{\widehat{\sigma}} E_A\,.$$

One endomorphism to rule them all?

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Answer: Essentially **yes!**

(See Benjamin's autumn-school lecture past Monday.)

One endomorphism to rule them all?

New <u>question</u>: Is computing *some* nonscalar endomorphism just as hard as finding $\varphi\colon E_0 \to E_A$?

<u>Answer:</u> Essentially **yes!**

(See Benjamin's autumn-school lecture past Monday.)

$\Longrightarrow$ Modulo minor details, soundness of SQIsign is equivalent to the hardness of the isogeny problem.

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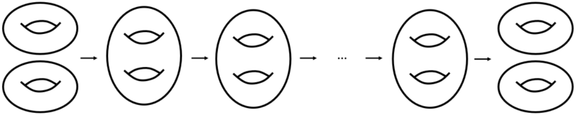
☺ Some newer SQIsign variants are much better in this regard!

⇝ See next talk.



Performance

SQIsign: Numbers

⚠️ These are from the round-1 submission to NISTPQC.
They will change very significantly in the coming months. 🚀

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sizes

parameter set	public keys	signatures
NIST-**I**	**64** bytes	**177** bytes
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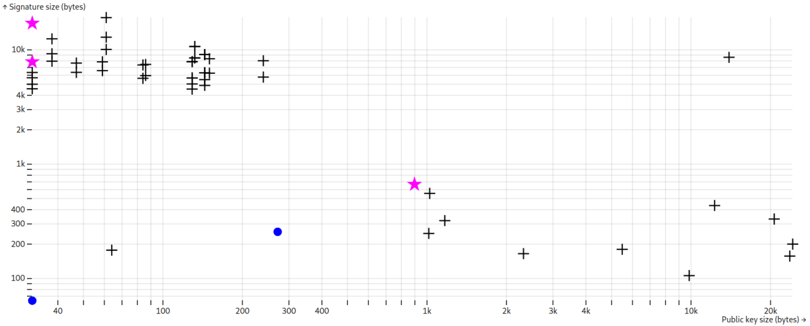
performance

Cycle counts for a *generic C implementation* running on an Intel *Ice Lake* CPU.
Optimizations are certainly possible and work in progress.

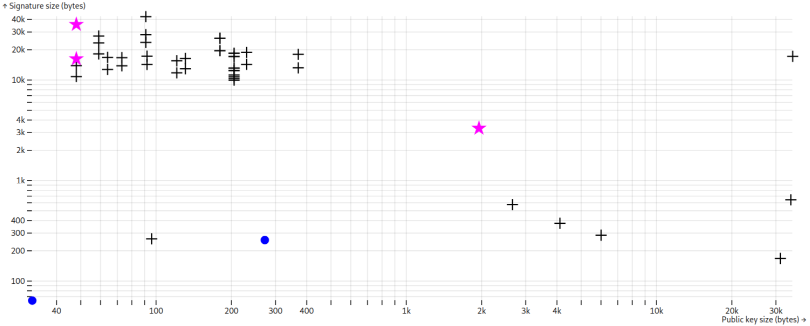
parameter set	keygen	signing	verifying
NIST-**I**	**3728** megacycles	**5779** megacycles	**108** megacycles
NIST-**III**	**23734** megacycles	**43760** megacycles	**654** megacycles
NIST-**V**	**91049** megacycles	**158544** megacycles	**2177** megacycles

SQIsign: Comparison (NIST level 1)



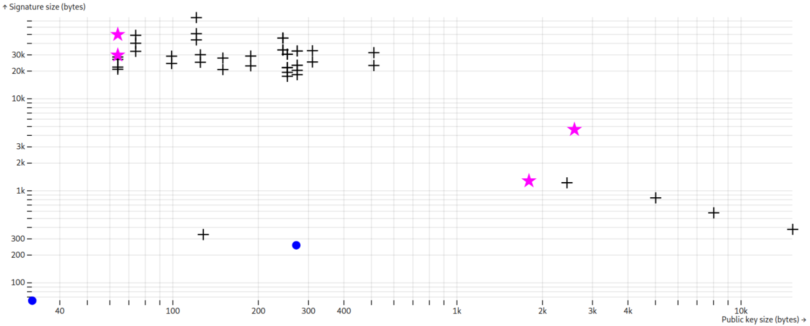
Source: `https://pqshield.github.io/nist-sigs-zoo`

SQIsign: Comparison (NIST level 3)



Source: `https://pqshield.github.io/nist-sigs-zoo`

SQIsign: Comparison (NIST level 5)



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Questions?

(Also feel free to email me: `lorenz@yx7.cc`)