SIKE不再

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- 2022: Never mind, it *is* broken.

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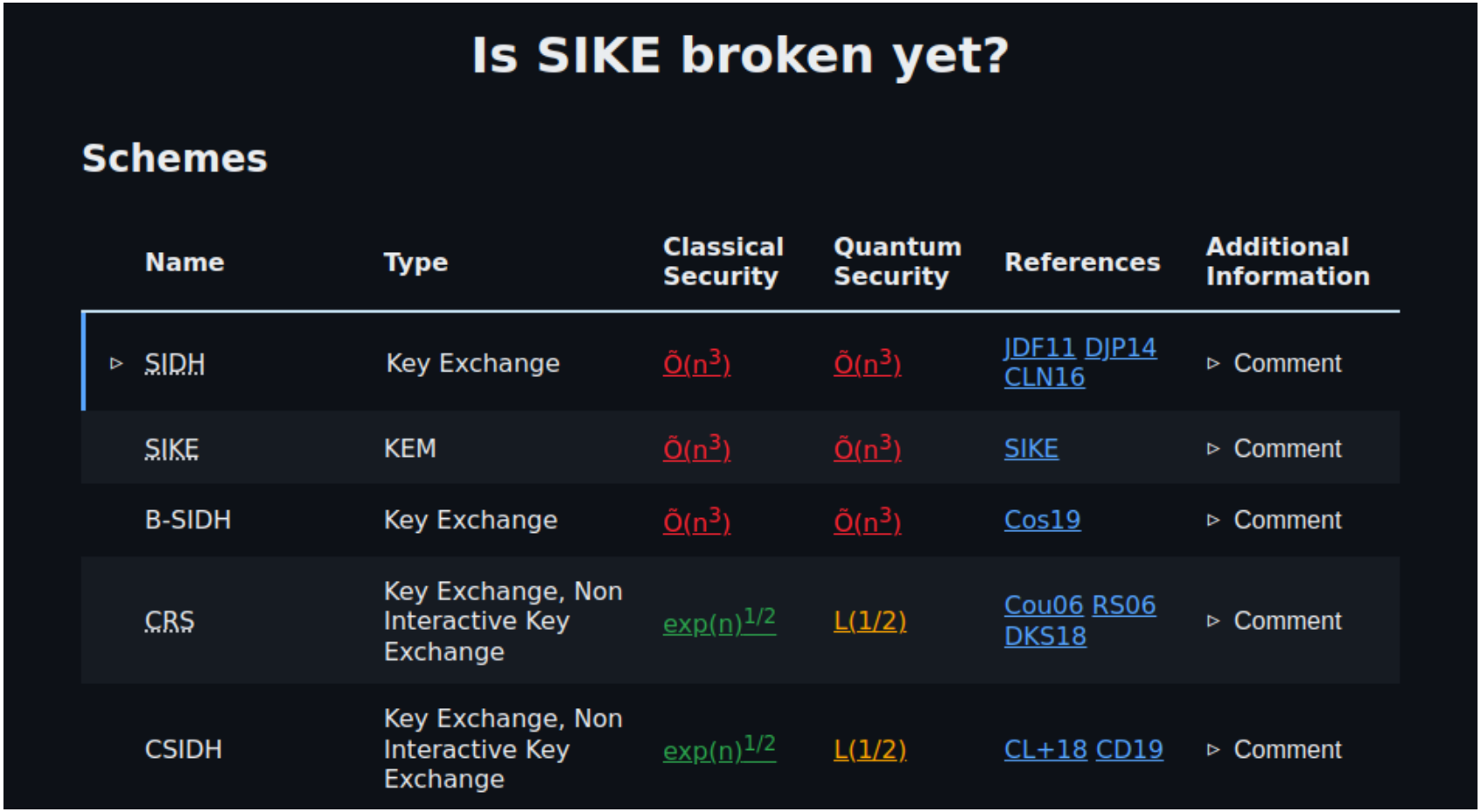
  Subsequent SageMath implementation (Pope–Oudompheng–...) takes < 2 hours.
- The attack relies on extremely cool mathematics.

The fallout

https://issikebrokenyet.github.io/

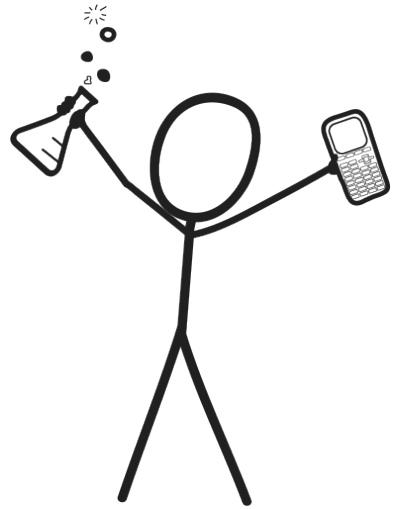


Is SIKE broken yet?

Schemes

Name	Type	Classical Security	Quantum Security	References	Additional Information
▹ SIDH	Key Exchange	$\tilde{O}(n^3)$	$\tilde{O}(n^3)$	JDF11 DJP14 CLN16	▹ Comment
SIKE	KEM	$\tilde{O}(n^3)$	$\tilde{O}(n^3)$	SIKE	▹ Comment
B-SIDH	Key Exchange	$\tilde{O}(n^3)$	$\tilde{O}(n^3)$	Cos19	▹ Comment
CRS	Key Exchange, Non Interactive Key Exchange	$\exp(n)^{1/2}$	L(1/2)	Cou06 RS06 DKS18	▹ Comment
CSIDH	Key Exchange, Non Interactive Key Exchange	$\exp(n)^{1/2}$	L(1/2)	CL+18 CD19	▹ Comment

Stand back!



We’re going to do math.

Isogenies

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- a group homomorphism.
- given by rational functions.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m] : E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \cong \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota : (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$ defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

The dual isogeny

Isogenies come in pairs.

Each isogeny $\varphi\colon E \to E'$ has a unique dual isogeny $\widehat{\varphi}\colon E' \to E$ characterized by $\widehat{\varphi} \circ \varphi = [\deg \varphi]$ and $\varphi \circ \widehat{\varphi} = [\deg \varphi]$.

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- Computing the dual is practically always efficient.

Isogeny kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G : E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

[1] (up to isomorphism of E')

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Vélu '71:
Formulas for computing E/G and evaluating φ_G at a point.

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Let’s talk about SIDH

Let's talk about ~~SIDH~~ the minimum set of knowledge about SIDH required to understand how to break it

SIDH: Bob's key-recovery problem

Fixed public data:

- Some large prime $p = 2^a 3^b - 1$.
- A particular starting curve E_0 defined over $\mathbb{F}_{p^2}$.
- Two points P_0, Q_0 on E_0 such that $E_0[2^a] = \langle P_0 \rangle + \langle Q_0 \rangle$.
  (In other words: a basis of the 2^a-torsion on E_0.)

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Bob's public key:

- The codomain curve E.
- The images $P = \varphi(P_0)$ and $Q = \varphi(Q_0)$ of P_0, Q_0.

Extending φ to the 2^a-torsion!

Recall:

- Given P_0, Q_0 and $P = \varphi(P_0)$ and $Q = \varphi(Q_0)$.
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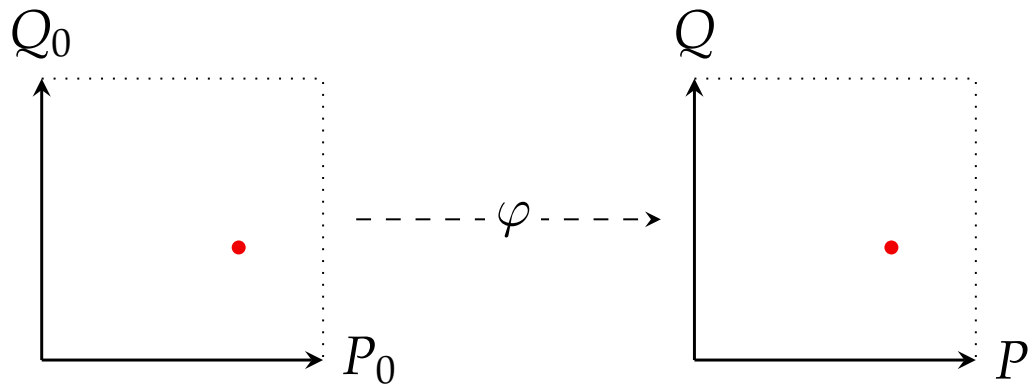

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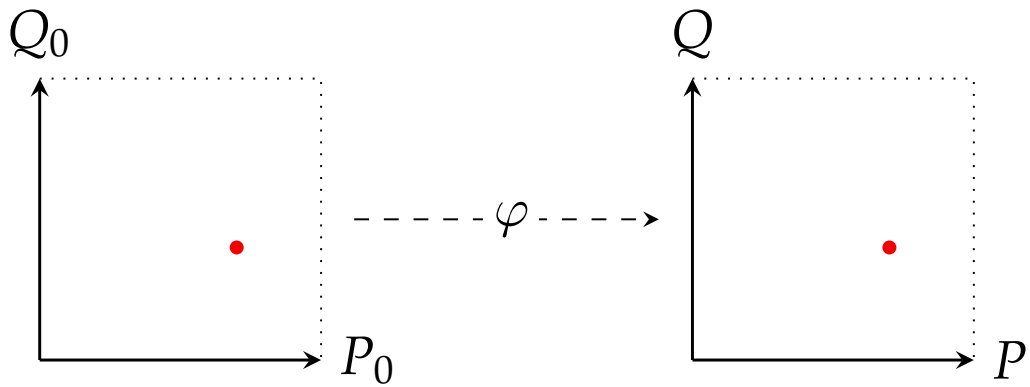

Evaluating φ at an arbitrary point $T \in E_0[2^a]$:

1. Decompose $T = [u]P_0 + [v]Q_0$ with $u, v \in \mathbb{Z}$.
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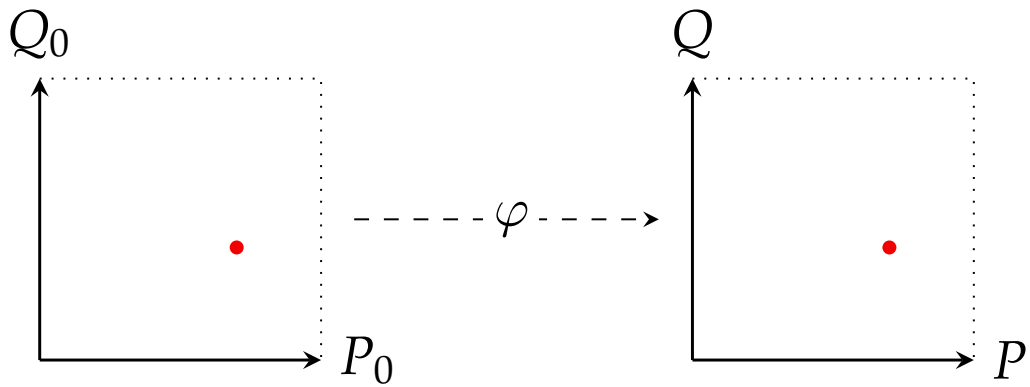

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$\Longrightarrow$ Effectively "have" the *restriction* $\varphi|_{E_0[2^a]}$ to the 2^a-torsion.

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Proof: Exercise :-)

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$\Longrightarrow$ can't learn anything about 3^b from 2^a using groups alone.

Attack idea: Interpolation?

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- No known algorithms for interpolating and decomposing at the same time.
  - Also unlikely to exist...

The attack

Gluing elliptic curves

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Computing isogenies of *products* of elliptic curves

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- Similar to elliptic curves in many ways:
  - Points form an abelian group.
  - Similar group structure, but more components.
  - Can define isogenies from kernel subgroups.
- Computing with surfaces explicitly is possible, but painful.
  Everyone works with Jacobians of genus-2 curves instead.

Splitting abelian surfaces

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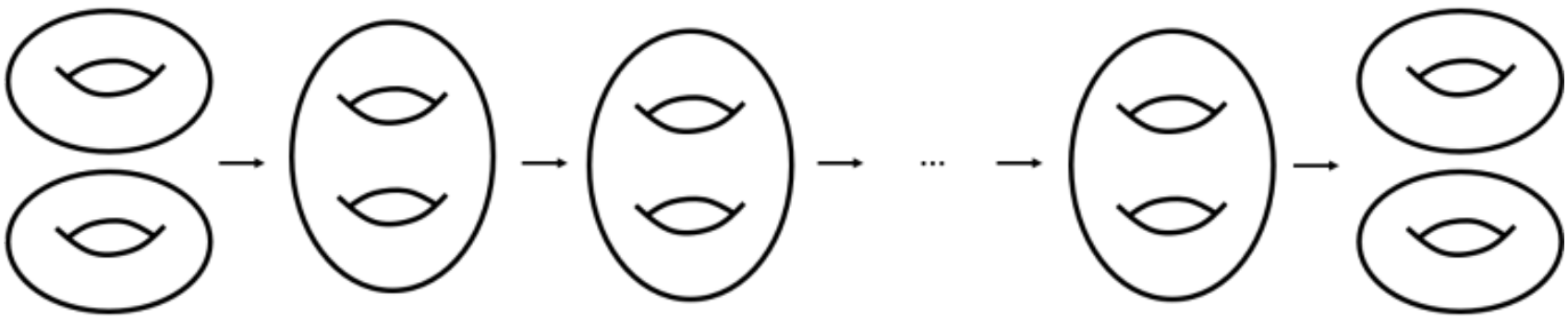

Picture: Castryck–Decru

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Suppose $\gcd(\deg\varphi, \deg\gamma) = 1$ and write $N = \deg\varphi + \deg\gamma$.
The subgroup

$$H := \left\{ \big(\varphi(T), \gamma(T)\big) \;\middle|\; T \in E_0[N] \right\}$$

is the kernel of an (N, N)-isogeny

$$\rho\colon E \times C \longrightarrow E_0 \times C'.$$

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Crucial observation: If $N \mid 2^a$, we can compute H from $\varphi|_{E_0[2^a]}$!

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"Tiny" problem: Ideally we'd want $N = 2^a$, so $\deg \gamma = 2^a - 3^b$.

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(Actually, the CD attack also involves a tiny bit of brute force for reasons I'll sweep under the rug. One consequence is that in the actual attack we'll have $\deg \gamma = 2^a - 3^{b-\beta}$ for a small β, but I'm pretending $\beta = 0$ for simplicity.)

The “shortcut” attack

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Better: The ρ from Kani has the very explicit description

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$$\rho(X, Y) = \big(\widehat{\varphi}(X) + \widehat{\gamma}(Y), \gamma'(X) - \varphi'(Y)\big)$$

where φ', γ' fit into the diagram:

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$\Longrightarrow$ We can simply evaluate $\widehat{\varphi}$ by computing $\rho(X, 0)$!

$\Longrightarrow$ As before: Evaluate on basis of $E[2^a]$ to find $\ker \varphi$.

Robert's generalization

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The endomorphism of $E_0^4 \times E^4$:

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-u & t & -w & v & 0 & \varphi & 0 & 0 \\
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- Core trick: *Embedding lemma*.
  This also has cool constructive applications! [ePrints 2022/1068 and 2022/1704]

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-u & t & -w & v & 0 & \varphi & 0 & 0 \\
-v & w & t & -u & 0 & 0 & \varphi & 0 \\
-w & -v & u & t & 0 & 0 & 0 & \varphi \\
-\widehat{\varphi} & 0 & 0 & 0 & t & -u & -v & -w \\
0 & -\widehat{\varphi} & 0 & 0 & u & t & w & -v \\
0 & 0 & -\widehat{\varphi} & 0 & v & -w & t & u \\
0 & 0 & 0 & -\widehat{\varphi} & w & v & -u & t
\end{pmatrix}$$

- Core trick: *Embedding lemma*.
  This also has cool constructive applications! [ePrints 2022/1068 and 2022/1704]

!! These are theoretical results. Currently totally impractical.

Closing remark

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Thank you!