Attacks and non-attacks on SIDH

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Largely based on "How to not break SIDH", which is joint work with Chloe Martindale.

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  Question for the next few dozens of minutes: Why?

Stand back!

We're going to do math.

Isogenies

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- given by rational functions.
- a group homomorphism.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m]\colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota\colon (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

Isogeny kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

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Vélu operates in the field where the points in G live.

$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$

$\rightsquigarrow$ this is (one reason) why we use supersingular curves!

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!! Evaluate φ_G as a chain of small-degree isogenies:
For $G \cong \mathbb{Z}/\ell^k$, set $\ker \psi_i := [\ell^{k-i}](\psi_{i-1} \circ \cdots \circ \psi_1)(G)$.

$$\begin{tikzcd} E \arrow[r, "\psi_1"] \arrow[rrrr, bend right, "\varphi_G"'] & E_1 \arrow[r, "\psi_2"] & \ldots \arrow[r, "\psi_{k-1}"] & E_{k-1} \arrow[r, "\psi_k"] & E/G \end{tikzcd}$$

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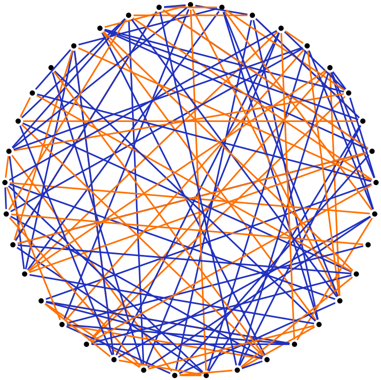
$\rightsquigarrow$ Complexity: $O(k^2 \cdot \ell)$. Exponentially smaller than ℓ^k!
"Optimal strategy" improves this to $O(k \log k \cdot \ell)$.

Isogeny graphs

- Graph view: Each ψ_i is a step in the ℓ-isogeny graph.



($q = 431^2$, degrees 2, 3)

Reminder:

SIDH

for those who missed David Jao's ECC talk 8 years ago $\ddot\smile$

SIDH: High-level view

E

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E $\quad$ A

B

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- Alice computes $\varphi_A : E \to E/A$; Bob computes $\varphi_B : E \to E/B$.
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- Alice <u>somehow</u> obtains $A' := \varphi_B(A)$. (Similar for Bob.)

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$$\begin{tikzcd} E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E/A \arrow[d, "\varphi_{B'}"] \\ E/B \arrow[r, "\varphi_{A'}"'] & E/\langle A, B\rangle \end{tikzcd}$$

- Alice & Bob pick secret subgroups A and B of E.
- Alice computes $\varphi_A \colon E \to E/A$; Bob computes $\varphi_B \colon E \to E/B$.
  (These isogenies correspond to walking in the isogeny graph.)
- Alice and Bob transmit the values E/A and E/B.
- Alice somehow obtains $A' := \varphi_B(A)$. (Similar for Bob.)
- They both compute the shared secret
  $$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

SIDH's auxiliary points

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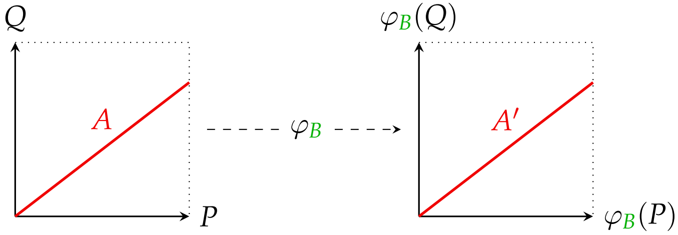

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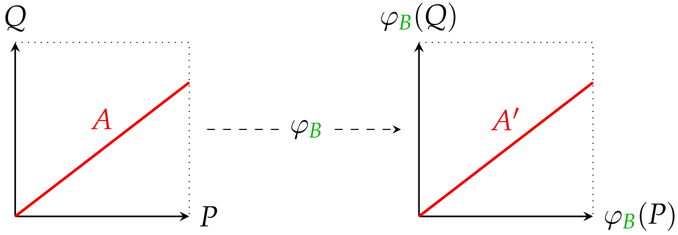

- Alice picks A as $\langle P + [a]Q \rangle$ for fixed public $P, Q \in E$.
- Bob includes $\varphi_B(P)$ and $\varphi_B(Q)$ in his public key.

$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$!

SIDH in one slide

Public parameters:

- a large prime $p = 2^n 3^m - 1$ and a supersingular $E/\mathbb{F}_p$
- bases (P_A, Q_A) and (P_B, Q_B) of $E[2^n]$ and $E[3^m]$

Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \dots 2^n - 1\}$		$b \xleftarrow{\text{random}} \{0 \dots 3^m - 1\}$
$A := \langle P_A + [a]Q_A \rangle$ compute $\varphi_A : E \to E/A$		$B := \langle P_B + [b]Q_B \rangle$ compute $\varphi_B : E \to E/B$
$E/A,\ \varphi_A(P_B),\ \varphi_A(Q_B)$		$E/B,\ \varphi_B(P_A),\ \varphi_B(Q_A)$
$A' := \langle \varphi_B(P_A) + [a]\varphi_B(Q_A) \rangle$ $s := j\big((E/B)/A'\big)$		$B' := \langle \varphi_A(P_B) + [b]\varphi_A(Q_B) \rangle$ $s := j\big((E/A)/B'\big)$

Disclaimer:

All of the following is “obvious” to experts.

Extra points: Information theory

- By linearity, the two points $\varphi_A(P_B), \varphi_A(Q_B)$ encode how φ_A acts on the whole 3^m-torsion.
- Note 3^m is smooth $\rightsquigarrow$ can evaluate φ_A on any $R \in E_0[3^m]$.

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Lemma. If two d-isogenies ϕ, ψ act the same on the m-torsion and $m^2 > 4d$, then $\phi = \psi$.

$\Longrightarrow$ Except for very unbalanced parameters, the public points uniquely determine the secret isogenies.

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$\rightsquigarrow$ can't even write down the result without decomposing into a sequence of smaller-degree maps.

- No known algorithms for interpolating and decomposing at the same time.
  - Also unlikely to exist...

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"[...] elliptic curves are as close to generic groups as it gets."

— me, 2018

(Exception: pairings, but those are "just" bilinear maps.)

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Read: An isogeny is uniquely defined by how it acts on sufficiently high ℓ^k-torsion.

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- This idea works slightly better for *endo*morphisms (characteristic polynomial constrains to ℓ^2 choices).

Extra points: Petit's endomorphisms (1)

- For typical SIDH parameters, we know endomorphisms ι, π of E_0 such that $\mathrm{End}(E_0) = \left\langle 1, \iota, \frac{\iota+\pi}{2}, \frac{1+\iota\pi}{2} \right\rangle$.

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- Going back and forth to E_0 yields endomorphisms of E_A:

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- Idea: Find $\tau \in R$ of degree $3^m r$; recover 3^m-part from known action; brute-force the $r$emaining part.
  $\Longrightarrow$ (details) $\Longrightarrow$ Recover φ_A.

Extra points: Petit's endomorphisms (2)

- Petit uses endomorphisms $\tau \in R$ of the form

  $$\tau = a + \varphi_A(b\iota + c\pi + d\iota\pi)\widehat{\varphi_A}\,,$$

  where $\deg \iota = 1$ and $\deg \pi = \deg \iota\pi = p$. Hence

  $$\deg \tau = a^2 + 2^{2n}b^2 + 2^{2n}pc^2 + 2^{2n}pd^2\,.$$

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$\Longrightarrow$ Petit's endomorphisms are not sufficiently petit-degree .

Auxiliary-points active attack [Galbraith–Petit–Shani–Ti]

- Recall: Bob sends $P' := \varphi_B(P)$ and $Q' := \varphi_B(Q)$ to Alice. She computes $A' = \langle P' + [a]Q' \rangle$ and, from that, obtains s.

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  $\Longrightarrow$ Bob learns the parity of a.
  Similarly, he can completely recover a in $O(n)$ queries.

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- Recall: Bob sends $P' := \varphi_B(P)$ and $Q' := \varphi_B(Q)$ to Alice. She computes $A' = \langle P' + [a]Q' \rangle$ and, from that, obtains s.
- Bob cheats and sends $Q'' := Q' + [2^{n-1}]P'$ instead of Q'. Alice computes $A'' = \langle P' + [a]Q'' \rangle$.

  If $a = 2u$: $[a]Q'' = [a]Q' + [u][2^n]P' = [a]Q'$.
  If $a = 2u+1$: $[a]Q'' = [a]Q' + [u][2^n]P' + [2^{n-1}]P' = [a]Q' + [2^{n-1}]P'$.

  $\Longrightarrow$ Bob learns the parity of a.
  Similarly, he can completely recover a in $O(n)$ queries.

Validating that Bob is honest is $\approx$ as hard as breaking SIDH.

$\Longrightarrow$ only usable with ephemeral keys or as a KEM *"SIKE"*.

Extra points: Summary

- Same problem all over the place:
  There seems to be no way to obtain *anything* from the given action-on-3^m-torsion except what's given.

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- Petit's approach cannot be expected to work for "real" (symmetric, two-party) SIDH.

  $\ddot\frown$

Interlude: How DH is SIDH?

Observation: SIDH has sometimes been marketed as "post-quantum Diffie–Hellman".

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Is this accurate?

- Not symmetric: Easily fixable, simply run two SIDH instances with opposite roles simultaneously. (This "invention" has been filed for patent in Canada...)
- Active attack: Not easily fixable; implies a significant lack of DH-ness!

...we’ll be right back after a short commercial break...

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[ˈsiːˌsaɪd]

...is an efficient commutative group action on an isogeny graph.

$\rightsquigarrow$ much closer to post-quantum Diffie–Hellman than SIDH $\ddot\smile$.

The pure isogeny problem

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Known solutions are generic: Graph walking, claw finding, ...
(These are all exponential-time, even quantumly.)

Equation solving?

Modular polynomials parameterize ℓ-isogenous j-invariants.
We are looking for an ℓ^n-isogeny between j_0 and j_n:

$$\Phi_\ell(j_0, X_1) = \Phi_\ell(X_1, X_2) = \Phi_\ell(X_2, X_3) = \dots$$
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Paper is still not online ¯_(ツ)_/¯, but it works exceptionally badly.

Weil restrictions?

"The Dream"

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3. Use Kuperberg's subexponential quantum algorithm for the abelian hidden-shift problem to find an isogeny ψ between the surfaces.
4. Hope we can solve the original problem better using ψ.
   - Can we always "unrestrict" back to $\mathbb{F}_{p^2}$ somehow?
   - Endomorphism-ring black magic?

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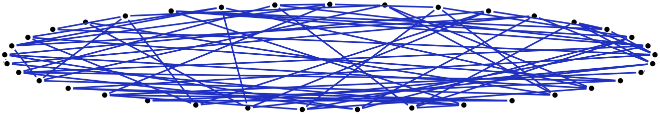
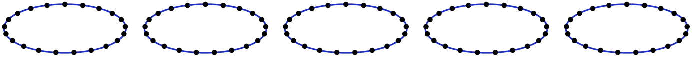


From a supersingular elliptic curve $E/\mathbb{F}_{p^2}$,

construct a superspecial abelian surface $A/\mathbb{F}_p$.



(Picture not to scale.)

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- Kuperberg can only work if the two abelian surfaces are in the same orbit... which is exponentially unlikely.
- There are more problems...
  - How to compute the group action in dimension 2?
  - Can we always lift back isogenies?

Lifting to $\mathbb{C}$?

"The Dream"

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3. Reduce Φ back modulo p to get $\varphi \colon E \to E'$.

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Well, none of this really seems to work:

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- Even given an endomorphism, lifting is prohibitively expensive if its degree is not small.
- Computing an isogeny over $\mathbb{C}$ still seems hard...

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Thank you!