# Isogenies in SageMath: Past, Present, Future

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# Part 0: *The Dream*

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sage: E0 = EllipticCurve(...)
sage: R = E0.endomorphism_ring()
sage: IA = R.random_ideal()
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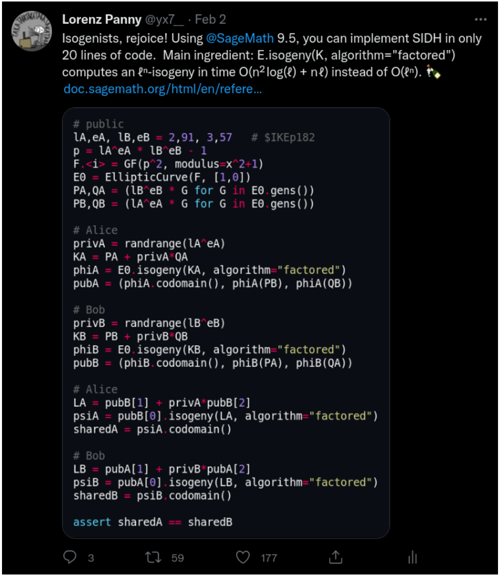
sage: I1 = R.ideal_from_isogeny(psi)
sage: S = EA.endomorphism_ring(IA)
sage: I2 = S.ideal_from_isogeny(phi)
sage: I = I2 * I1 * IA.conjugate()
sage: J = I.equivalent_smooth_ideal()
sage: response = J.isogeny()
```

# The rude awakening



Part 1: *Past*

Part 2: *Present*

Part 3: *Future*

# The past

Situation $\approx 2020$:

- `EllipticCurveIsogeny`: implementing Vélu and Kohel.
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- Fair share of bugs!

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def _discrete_log(self,x):
    ...
    # EVEN DUMBER IMPLEMENTATION!
    ...
    u = [y for y in self.list() if y.element() == x]
    if len(u) == 0: raise TypeError("Not in group")
    if len(u) > 1: raise NotImplementedError
    return u[0].vector()
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Sage $\geq$ 9.6:

```
sage: a,b = E.abelian_group().discrete_log(R)
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Composing two `EllipticCurveIsogeny` objects "works",
but is not overly useful:

```
sage: phi = phi2 * phi1
sage: type(phi)
<class 'sage.categories.map.FormalCompositeMap'>
sage: phi.degree()
AttributeError: ...
sage: phi.rational_maps()
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Addition of isogenies is not implemented at all:

```
sage: phi + phi
TypeError:
    unsupported operand parent(s) for +: 'Set of morphisms ...'
```

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- Composing with isomorphisms rather awkward.

```
sage: tau = E.automorphisms()[1]
sage: phi = E.isogeny(...)
sage: psi = phi * tau                          # nope
TypeError: self (=Isogeny of degree ...) domain
    must equal right (=Generic endomorphism of Abelian group
    of points on Elliptic Curve defined by ...) codomain
sage: phi.set_pre_isomorphism(tau)  # okay; in-place
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- Almost all of the usual isogeny methods missing: `.degree()`, `.rational_maps()`, `.formal()`, ...

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- `EllipticCurve_finite_field` has `.frobenius()`, but it just returns an element of a quadratic field:

```
sage: E = EllipticCurve(GF(101), [5,5])
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Order in Number Field in phi
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- Inseparable isogenies are actually irrepresentable.
  (This includes the Frobenius endomorphism in the supersingular case!)
- No algorithm for traces or degrees, or anything else.
  Crucial tool for computing the structure of an endomorphism (sub)ring!

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## Part 2: *Present*

Part 3: *Future*

# The present: Unified parent class for isogenies

- Sage $\geq 9.5$: Common class `EllipticCurveHom` for `EllipticCurveIsogeny`, `WeierstrassIsomorphism`, and other (new) types of elliptic-curve morphisms.

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- Compose any two isogenies using the `*` operator.
  !! This is currently opt-in for some type combinations. Use `EllipticCurveHom_composite.make_default()`. Soon™ default.

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- Patch for quasilinear strategy is ready, but stuck.

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  - Similarly, we can represent $\pi_r\colon E \to E^{(p^k)}$, $(x, y) \mapsto (x^{p^k}, y^{p^k})$ simply as $(E, k)$, enriched in the same way.
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- Simply evaluate on generators of large enough subgroup. (May require taking an extension of degree $O(\log d)$.)
- This is essentially a version of *polynomial identity testing*, optimized for maps defining a group homomorphism.

# The present: $\sqrt{\text{élu}}$

Sage $\geq 9.7$ (released *eergisteren*!):

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sage: l = 10000019
sage: p = 40*l - 1
sage: E = EllipticCurve(GF(p), [1,0])
sage: P = (p+1)//l * E.gens()[0]
sage: E.isogeny(P, algorithm='velusqrt')
Elliptic-curve isogeny (using √élu) of degree 10000019:
  From: Elliptic Curve defined by y^2 = x^3 + x
          over Finite Field of size 400000759
  To:   Elliptic Curve defined by
          y^2 = x^3 + 88879239*x + 195338414
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- Vélu's formulas take about 8 minutes for the same isogeny.
- Speedup is even more significant as the degree grows.

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**Q**: What does "simplify" mean? Expand everything? Group common summands? **How to apply the distributive law?**

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Good support for sums and compositions of isogenies
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For usability/flexibility:

- Important to support endomorphism subrings for working with oriented supersingular curves (in particular: CSIDH), or when having only a non-full-index subring.
- Important to render existing algebraic tools for orders in {quadratic fields, quaternion algebras} easily applicable to endomorphism rings.

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- To figure out abstract version of a concrete morphism, compute trace pairings with the defining morphisms.

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- Quadratic case requires ideals of non-maximal quadratic orders. Work in progress.

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  Basic algorithm: Brute-force random cycles in some isogeny graph; compute relations (trace pairings); repeat until full-rank and full-index.
  This is exponential-time.
- Quadratic case: Isogeny volcano walking.
  Beautiful theory; see for instance Sutherland's "Isogeny Volcanoes".
  This is efficient in many cases! [Preliminary implementation exists.]

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Oh, and also: **Genus 2**, for reasons. LOL

Part 1: *Past*

Part 2: *Present*

Part 3: *Future*

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You can help!

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- Make sure to run the most recent version: Lots of speed improvements for elliptic curves and isogenies.