Isogenies I & II

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Please ask me anything!

Diffie–Hellman key exchange '76

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today elliptic curves)
- an element $g \in G$ of prime order q

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<u>Alice</u>	<u>public</u>	<u>Bob</u>
$a \xleftarrow{\text{random}} \{0 \ldots q-1\}$		$b \xleftarrow{\text{random}} \{0 \ldots q-1\}$
g^a		g^b
$s := (g^b)^a$		$s := (g^a)^b$

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Fundamental reason this works: $\cdot^a$ and $\cdot^b$ are commutative!

Diffie–Hellman: Bob vs. Eve

<u>Bob</u>

1. Set $t \leftarrow g$.
2. Set $t \leftarrow t \cdot g$.
3. Set $t \leftarrow t \cdot g$.
4. Set $t \leftarrow t \cdot g$.

...

$b-2$. Set $t \leftarrow t \cdot g$.

$b-1$. Set $t \leftarrow t \cdot g$.

b. Publish $B \leftarrow t \cdot g$.

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Is this a good idea?

Diffie–Hellman: Bob vs. Eve

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$b{-}1$. Set $t \leftarrow t \cdot g$.

b. Publish $B \leftarrow t \cdot g$.

Attacker Eve

1. Set $t \leftarrow g$. If $t = B$ return 1.
2. Set $t \leftarrow t \cdot g$. If $t = B$ return 2.
3. Set $t \leftarrow t \cdot g$. If $t = B$ return 3.
4. Set $t \leftarrow t \cdot g$. If $t = B$ return 3.

...

$b{-}2$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b{-}2$.

$b{-}1$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b{-}1$.

b. Set $t \leftarrow t \cdot g$. If $t = B$ return b.

$b{+}1$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b + 1$.

$b{+}2$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b + 2$.

...

Diffie–Hellman: Bob vs. Eve

Bob	Attacker Eve
1. Set $t \leftarrow g$.	1. Set $t \leftarrow g$. If $t = B$ return 1.
2. Set $t \leftarrow t \cdot g$.	2. Set $t \leftarrow t \cdot g$. If $t = B$ return 2.
3. Set $t \leftarrow t \cdot g$.	3. Set $t \leftarrow t \cdot g$. If $t = B$ return 3.
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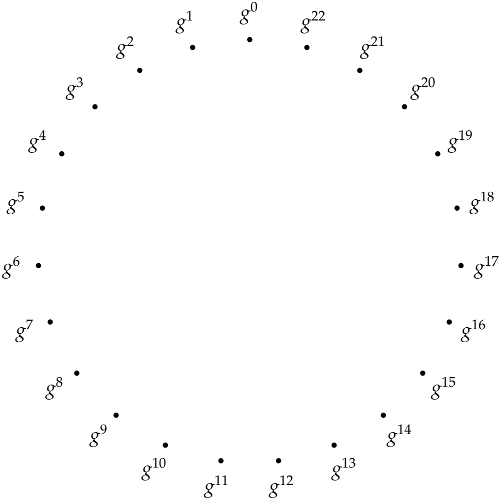
	$b+2$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b+2$.
	...

Effort for both: $O(\#G)$. Bob needs to be smarter.

(This attacker is also kind of dumb, but that doesn't matter for my point here.)


g^0
g^1
g^2
g^3
g^4
g^5
g^6
g^7
g^8
g^9
g^{10}
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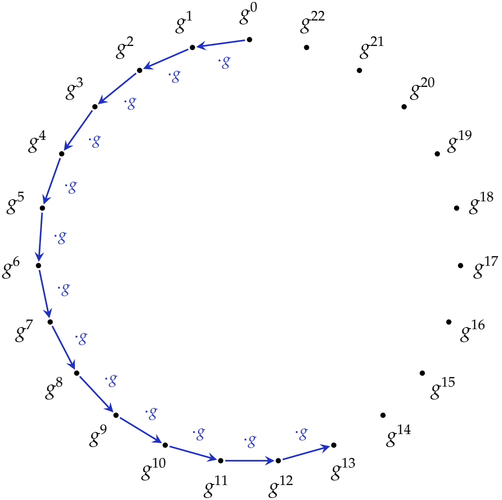
g^{12}
g^{13}
g^{14}
g^{15}
g^{16}
g^{17}
g^{18}
g^{19}
g^{20}
g^{21}
g^{22}

multiply



g^0
g^1
g^2
g^3
g^4
g^5
g^6
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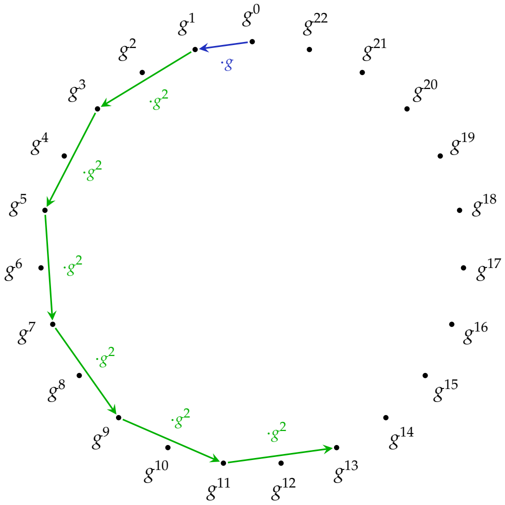
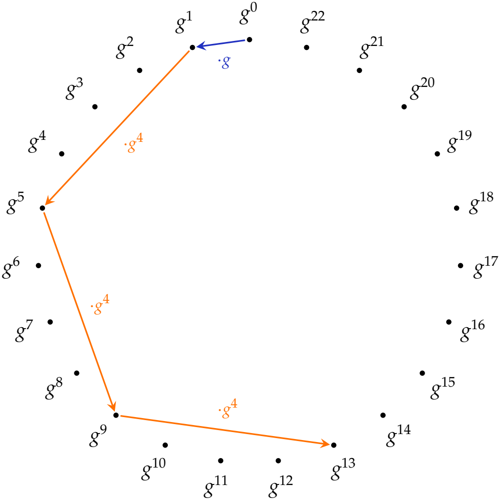
g^9
g^{10}
g^{11}
g^{12}
g^{13}
g^{14}
g^{15}
g^{16}
g^{17}
g^{18}
g^{19}
g^{20}
g^{21}
g^{22}
$\cdot g$

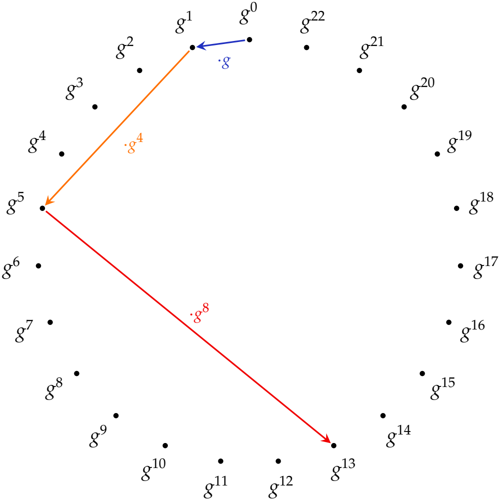
Square-and-multiply



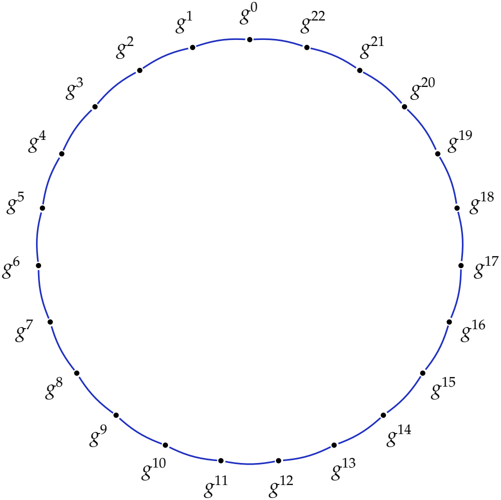
Square-and-multiply-and-square-and-multiply



Square-and-multiply-and-square-and-multiply-and-squ



Square-and-multiply as graphs



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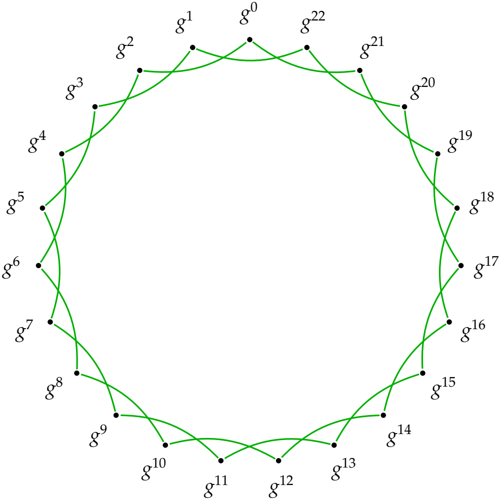

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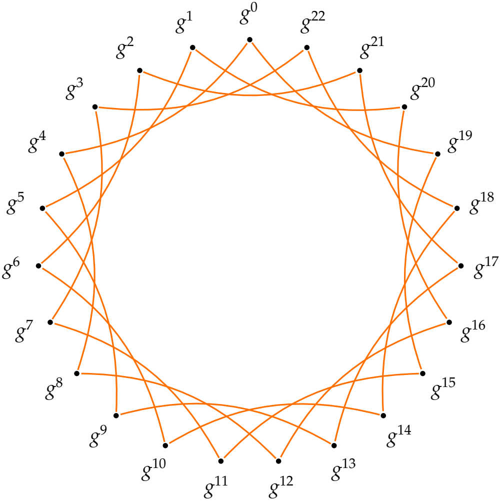

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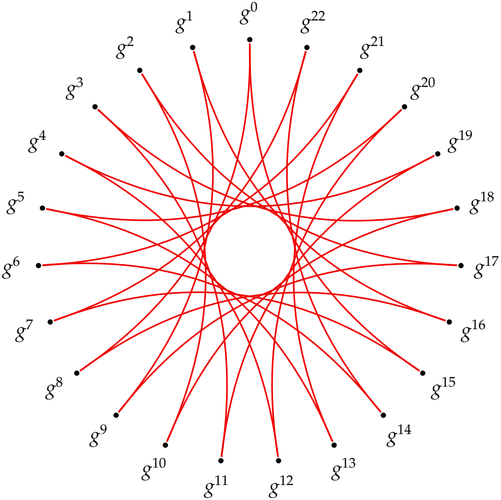

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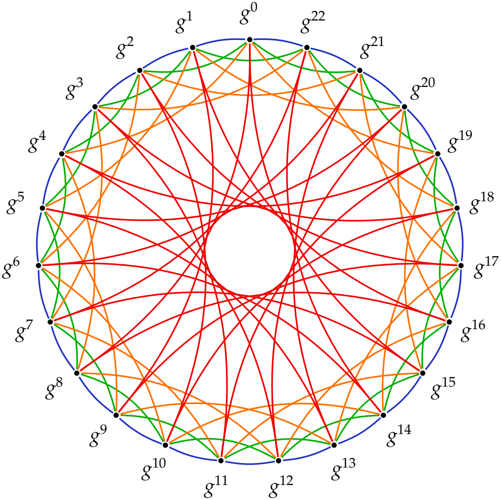

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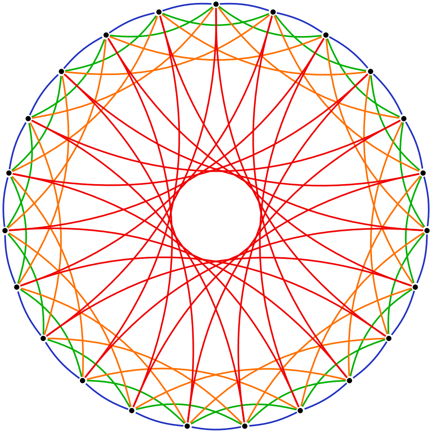

Fast mixing: paths of length log(# nodes) to everywhere.

With square-and-multiply, applying b takes $\Theta(\log \#G)$.

For well-chosen groups, recovering b takes $\Theta(\sqrt{\#G})$.

$\rightsquigarrow$ Exponential separation!

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...and they lived happily ever after?

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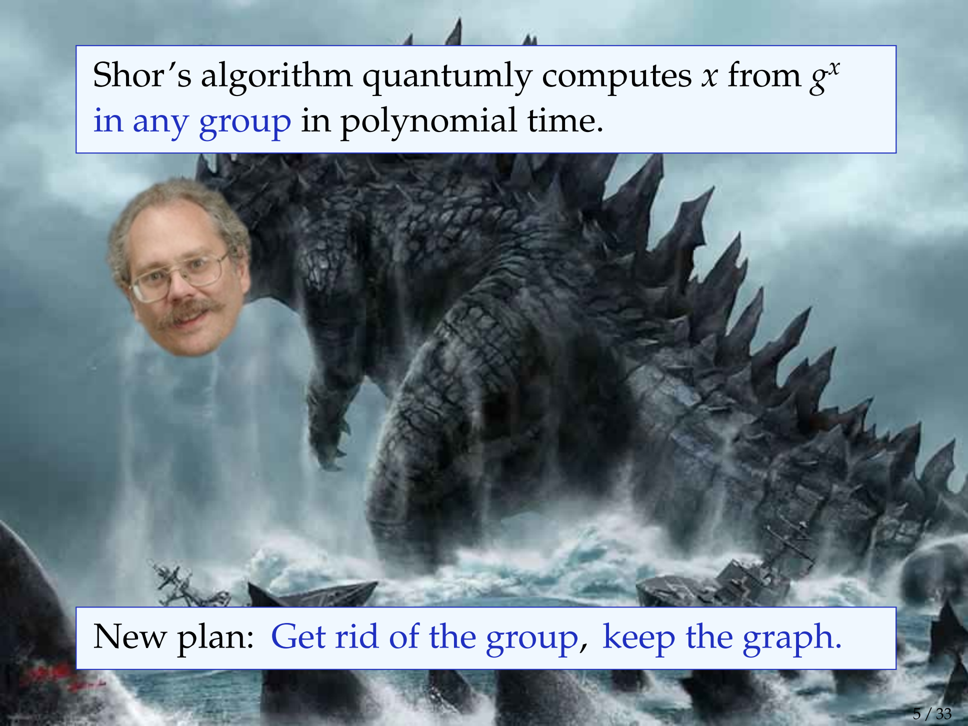

New plan: Get rid of the group, keep the graph.

Big picture

- Isogenies are a source of exponentially-sized graphs.

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It is easy to construct graphs that satisfy *almost* all of these —
not enough for crypto!

Upshot

Isogenies give rise to

‘post-quantum Diffie–Hellman’.

(and more!)

Slightly smaller picture

- Isogenies are well-behaved maps between elliptic curves.

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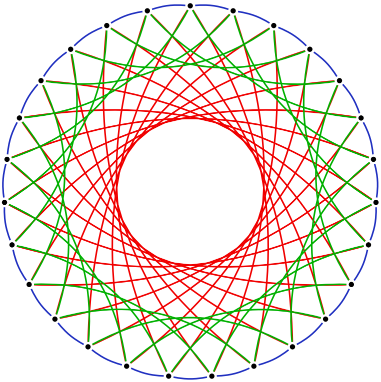
- Isogenies are well-behaved maps between elliptic curves.

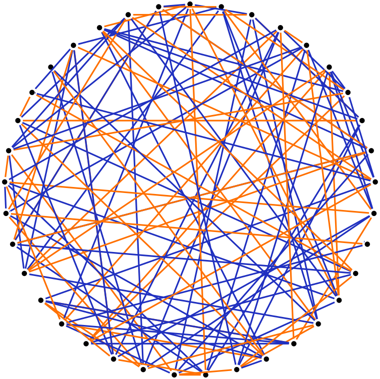
$\rightsquigarrow$ Isogeny graph: Nodes are curves, edges are isogenies.

(We usually care about subgraphs with certain properties.)

The beauty and the beast

Components of well-chosen isogeny graphs look like this:





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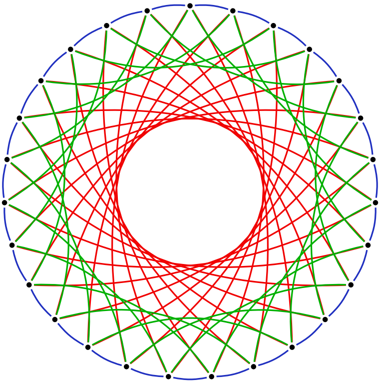

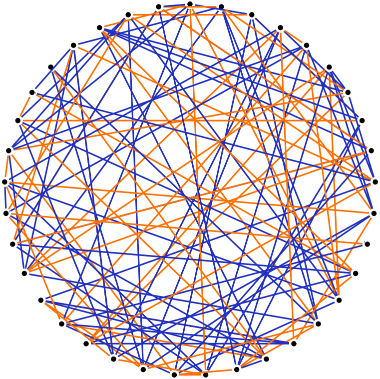

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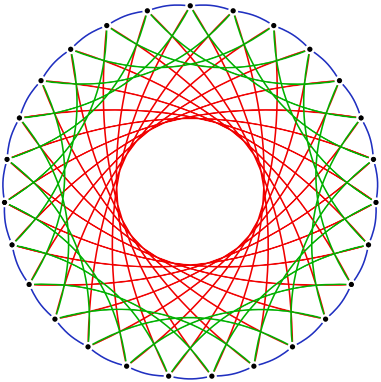
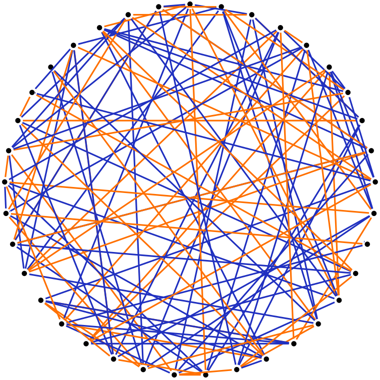
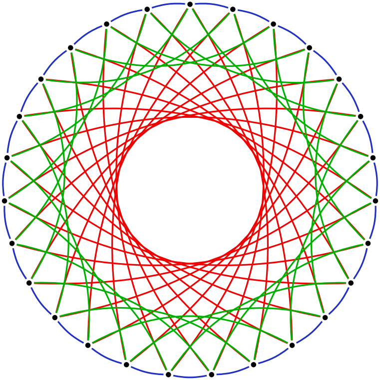




Which of these is good for crypto? **Both.**

The beauty and the beast

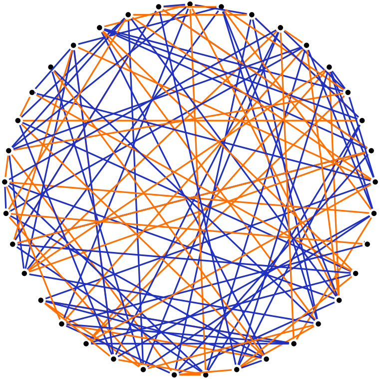
At this time, there are two distinct families of systems:



$q = p$

CSIDH [ˈsiːˌsaɪd]

`https://csidh.isogeny.org`



$q = p^2$

SIDH

`https://sike.org`

CSIDH [ˈsiːˌsaɪd]

(Castryck, Lange, Martindale, Panny, Renes; 2018)

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- Small keys: starts at 64 bytes.*
- Competitive speed: $\approx$ 55 ms / full key exchange.* (Skylake)
- Flexible: compatible with 0-RTT protocols such as QUIC; yields signatures, (pre-quantum) VDFs, etc.

---

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Stand back!

We’re going to do math.

Math slide #1: Elliptic curves *(nodes)*

An elliptic curve (modulo details) is given by an equation

$$E\colon\; y^2 = x^3 + ax + b.$$

A point on E is a solution to this equation *or* the 'fake' point ∞.

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A point on E is a solution to this equation *or* the 'fake' point ∞.

E is an abelian group: we can 'add' and 'subtract' points.

- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

$$\left(\lambda^2 - x_1 - x_2,\ \lambda(2x_1 + x_2 - \lambda^2) - y_1\right)$$

do **not** remember these formulas!

where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + a}{2y_1}$ otherwise.

Math slide #2: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$

- given by rational functions
- that is a group homomorphism.

The degree of a separable isogeny is the size of its kernel.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m]\colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota\colon (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}\,.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

CSIDH in one slide

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- Make sure $p = 4 \cdot \ell_1 \cdots \ell_n - 1$ is prime.

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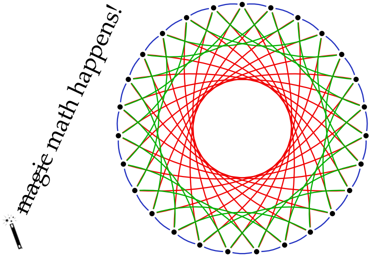

$p = 419$
$\ell_1 = 3$
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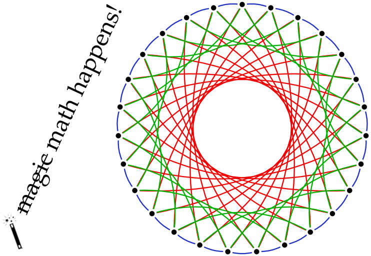

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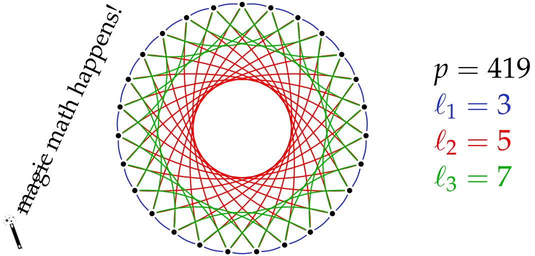

- Walking 'left' and 'right' on any ℓ_i-subgraph is efficient.
- We can represent $E \in X$ as a single coefficient $A \in \mathbb{F}_p$.

Walking in the CSIDH graph

Taking a 'positive' step on the ℓ_i-subgraph.

1. Find a point $(x, y) \in E$ of order ℓ_i with $x, y \in \mathbb{F}_p$.
   This uses scalar multiplication by $(p+1)/\ell_i$.
2. Compute the isogeny with kernel $\langle (x, y) \rangle$ (see next slide).

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Taking a 'negative' step on the ℓ_i-subgraph.

1. Find a point $(x, y) \in E$ of order ℓ_i with $x \in \mathbb{F}_p$ <u>but $y \notin \mathbb{F}_p$</u>.
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2. Compute the isogeny with kernel $\langle (x, y) \rangle$ (see next slide).

Upshot: With 'x-only arithmetic' everything happens over $\mathbb{F}_p$.
$\implies$ Efficient to implement!

Math slide #3: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is called E/G. ($\approx$ quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

[1](up to isomorphism of E')

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Vélu '71:
Formulas for computing E/G and evaluating φ_G at a point.

Complexity: $\Theta(\#G)$ $\rightsquigarrow$ only suitable for small degrees.

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Math slide #3: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is called E/G. (≈ quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

Vélu '71:
Formulas for computing E/G and evaluating φ_G at a point.

Complexity: $\Theta(\#G)$ $\rightsquigarrow$ only suitable for small degrees.

Vélu operates in the field where the points in G live.

$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$

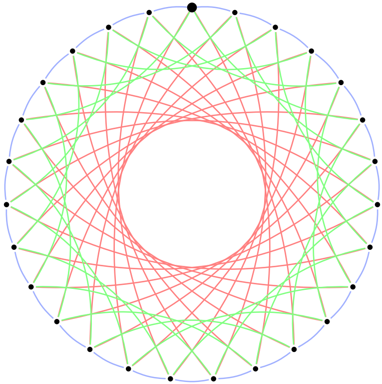
$\rightsquigarrow$ this is why we use special p and curves with $p+1$ points!

[1](up to isomorphism of E')

CSIDH key exchange

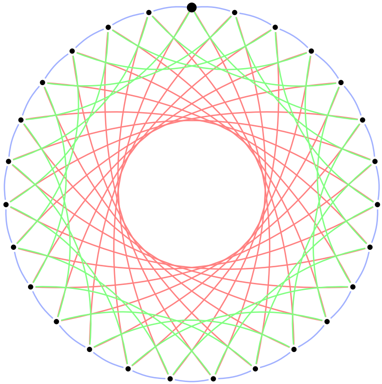
Alice

[+, +, −, −]



Bob

[−, +, −, −]



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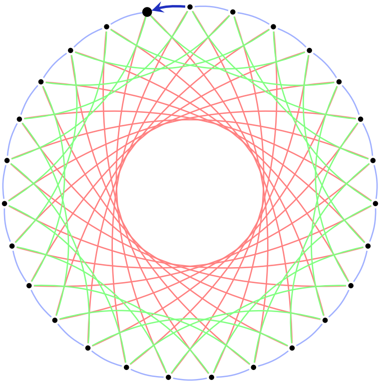

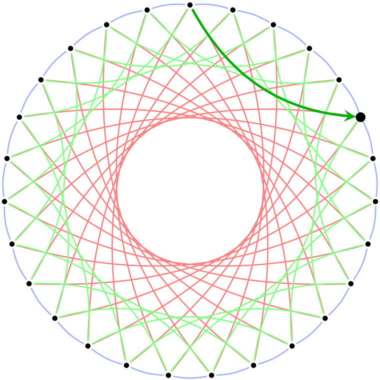

CSIDH key exchange

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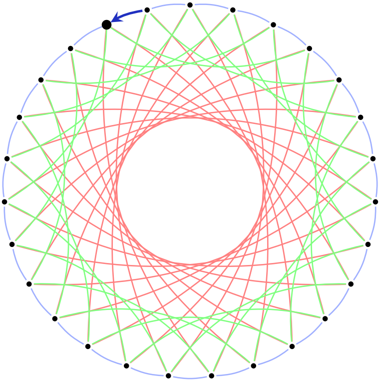
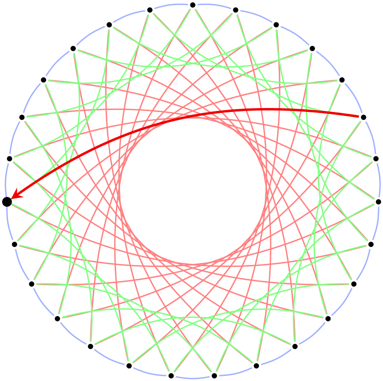
$[\textcolor{blue}{+}, \underset{\uparrow}{\textcolor{blue}{+}}, \textcolor{red}{-}, \textcolor{green}{-}]$



Bob

$[\textcolor{green}{-}, \underset{\uparrow}{\textcolor{red}{+}}, \textcolor{green}{-}, \textcolor{blue}{-}]$



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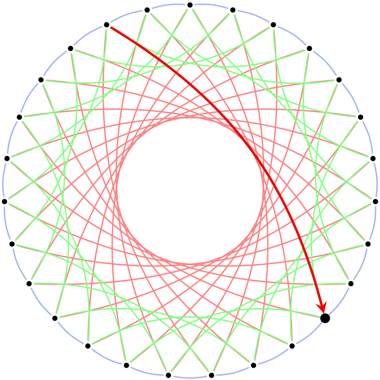

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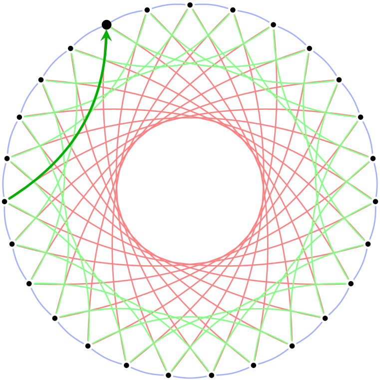

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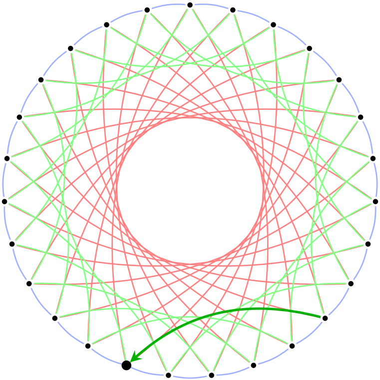

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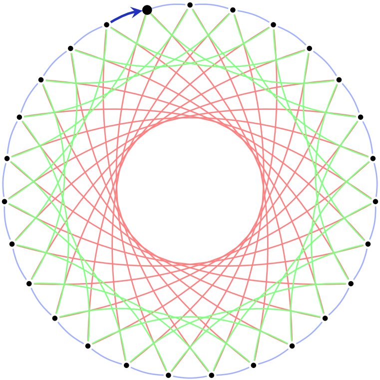

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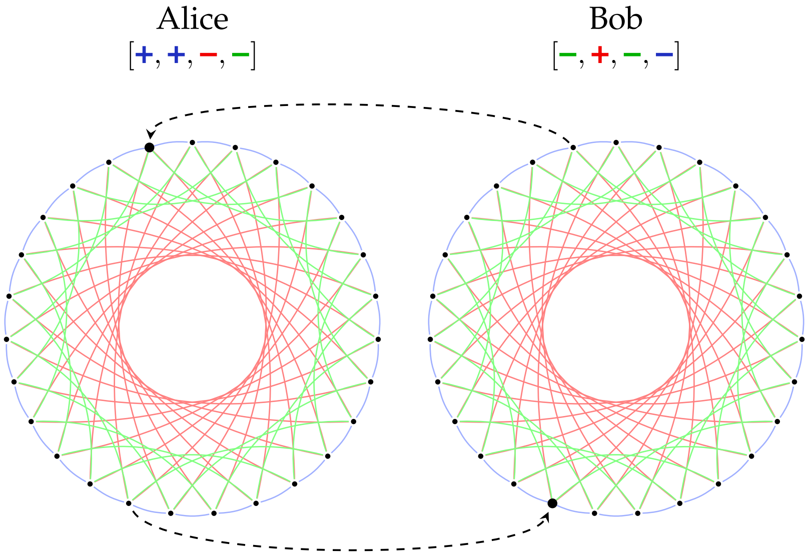

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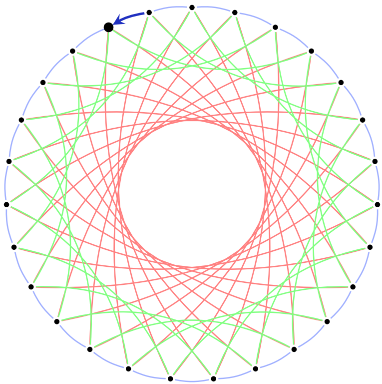

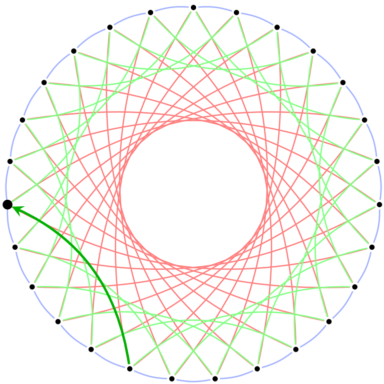

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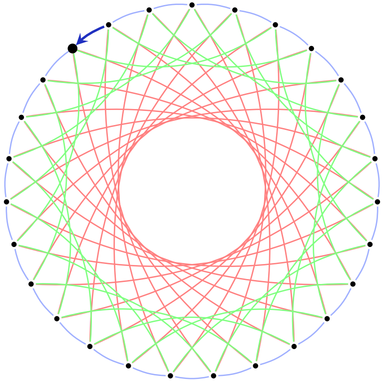

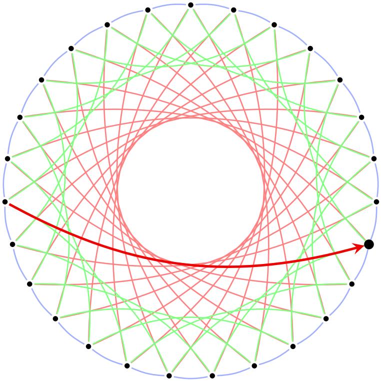

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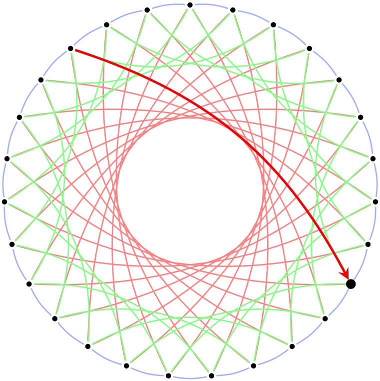

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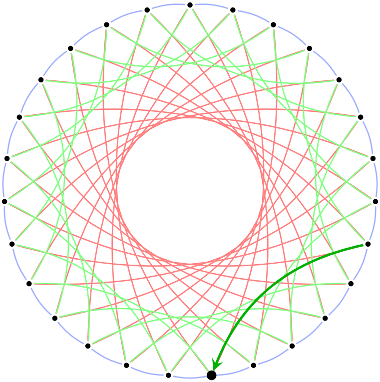

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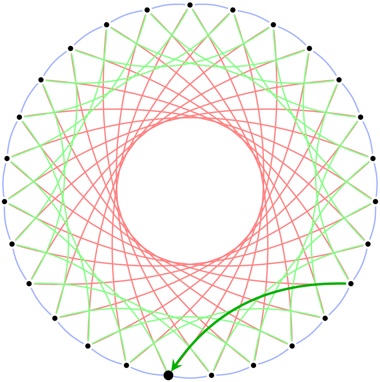

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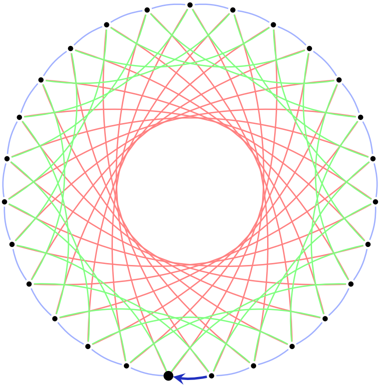

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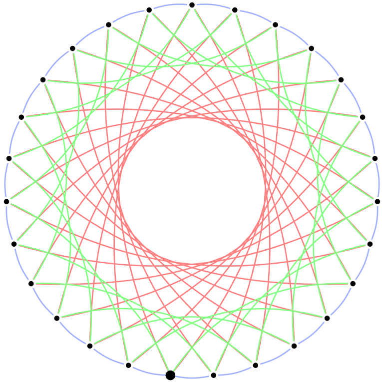

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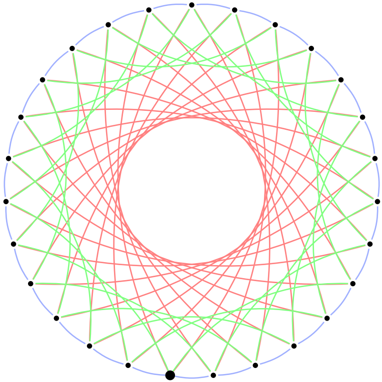

Has anyone seen my group action?

"CSIDH: an efficient post-quantum
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Cycles are compatible: [right then left] = [left then right]
$\rightsquigarrow$ only need to keep track of total step counts for each ℓ_i.

Example: [**+**, **+**, **−**, **−**, **−**, **+**, **−**, **−**] just becomes $(+1, \quad 0, -3) \in \mathbb{Z}^3$.

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There is a group action of $(\mathbb{Z}^n, +)$ on our set of curves X!

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Many paths are 'useless'. *Fun fact:* Quotienting out trivial actions yields the ideal-class group $\mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$.

Cryptographic group actions

Like in the CSIDH example, we *generally* get a DH-like key exchange from a commutative group action $G \times S \to S$:

Alice	public	Bob
$a \xleftarrow{\text{random}} G$		$b \xleftarrow{\text{random}} G$
$a * s$		$b * s$
$\mathit{key} := a * (b * s)$		$\mathit{key} := b * (a * s)$

Why no Shor?

Recall from Dan's talk:

Shor computes α from $h = g^\alpha$ by finding the kernel of the map

$$f\colon\ \mathbb{Z}^2 \to G,\ (x, y) \mapsto g^x \cdot h^y$$

For general group actions, we cannot compose $a * s$ and $b * s$!

Security of CSIDH

Core problem:
Given $E, E' \in X$, find a smooth-degree isogeny $E \to E'$.

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Solving abelian hidden shift breaks CSIDH.

$\rightsquigarrow$ quantum subexponential attack (Kuperberg's algorithm).

CSIDH vs. Kuperberg

Kuperberg's algorithm consists of two components:

1. Evaluate the group action many times. ('oracle calls')
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  (Is one qubit operation $\approx$ one bit operation? a hundred? millions?)

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$\Longrightarrow$ It is still rather unclear how to choose CSIDH parameters.

...but all known attacks cost $\exp\big((\log p)^{1/2+o(1)}\big)$!

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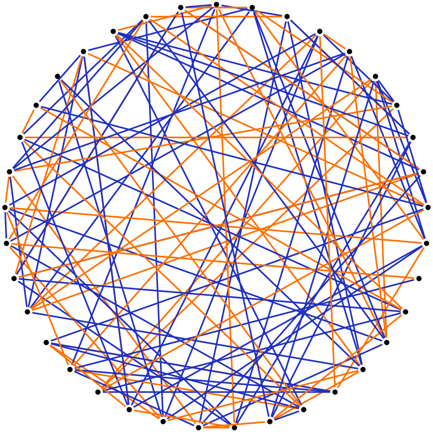
- Problem: also no more intrinsic sense of direction.

"It all bloody looks the same!" — a famous isogeny cryptographer

$\rightsquigarrow$ need extra information to let Alice & Bob's walks commute.



Now: SIDH (Jao, De Feo; 2011)

(...whose name doesn't allow for nice pictures of beaches...)

Wikipedia about SIDH...

"While several steps of SIDH involve complex isogeny calculations, the overall flow of SIDH for parties A and B is straightforward for those familiar with a Diffie–Hellman key exchange or its elliptic curve variant. [...]

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Setup.

1. A prime of the form $p = w_A^{e_A} \cdot w_B^{e_B} \cdot f \pm 1$.
2. A supersingular elliptic curve E over $\mathbb{F}_{p^2}$.
3. Fixed elliptic points P_A, Q_A, P_B, Q_B on E.
4. The order of P_A and Q_A is $(w_A)^{e_A}$.
5. The order of P_B and Q_B is $(w_B)^{e_B}$.

Key exchange. [...]

1A. A generates two random integers $m_A, n_A < (w_A)^{e_A}$.
2A. A generates $R_A := m_A \cdot (P_A) + n_A \cdot (Q_A)$.
3A. A uses the point R_A to create an isogeny mapping $\phi_A : E \to E_A$ and curve E_A isogenous to E.
4A. A applies ϕ_A to P_B and Q_B to form two points on E_A : $\phi_A(P_B)$ and $\phi_A(Q_B)$.
5A. A sends to B E_A, $\phi_A(P_B)$, and $\phi_A(Q_B)$.
1B–4B. Same as A1 through A4, but with A and B subscripts swapped.
5B. B sends to A E_B, $\phi_B(P_A)$, and $\phi_B(Q_A)$.
6A. A has $m_A, n_A, \phi_B(P_A)$, and $\phi_B(Q_A)$ and forms $S_{BA} := m_A(\phi_B(P_A)) + n_A(\phi_B(Q_A))$.
7A. A uses S_{BA} to create an isogeny mapping ψ_{BA}.
8A. A uses ψ_{BA} to create an elliptic curve E_{BA} which is isogenous to E.
9A. A computes $K :=$ j-invariant (j_{BA}) of the curve E_{BA}.
6B. Similarly, B has $m_B, n_B, \phi_A(P_B)$, and $\phi_A(Q_B)$ and forms $S_{AB} = m_B(\phi_A(P_B)) + n_B(\phi_A(Q_B))$.
7B. B uses S_{AB} to create an isogeny mapping ψ_{AB}.
8B. B uses ψ_{AB} to create an elliptic curve E_{AB} which is isogenous to Ek
9B. B computes $K :=$ j-invariant (j_{AB}) of the curve E_{AB}.

The curves E_{AB} and E_{BA} are guaranteed to have the same j-invariant."

SIDH: High-level view

$$\begin{tikzcd}
E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E/A \arrow[d, "\varphi_{B'}"] \\
E/B \arrow[r, "\varphi_{A'}"'] & E/\langle A, B\rangle
\end{tikzcd}$$

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- Alice & Bob pick secret subgroups A and B of E.
- Alice computes $\varphi_A : E \to E/A$; Bob computes $\varphi_B : E \to E/B$.
  (These isogenies correspond to walking on the isogeny graph.)

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- They both compute the shared secret
  $$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

SIDH's auxiliary points

Previous slide: "Alice somehow obtains $A' := \varphi_B(A)$."

Alice knows only A, Bob knows only φ_B. Hm.

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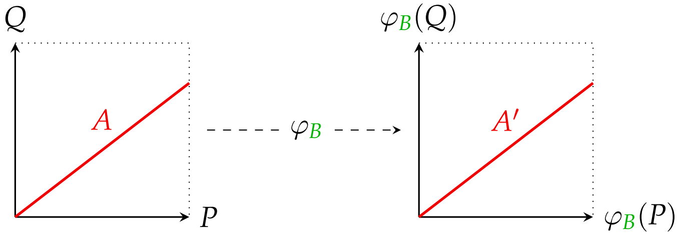
Alice knows only A, Bob knows only φ_B. Hm.

Solution: φ_B is a group homomorphism!

- Alice picks A as $\langle P + [a]Q \rangle$ for fixed public $P, Q \in E$.
- Bob includes $\varphi_B(P)$ and $\varphi_B(Q)$ in his public key.

$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$!



SIDH in one slide

Public parameters:

- a large prime $p = 2^n 3^m - 1$ and a supersingular $E/\mathbb{F}_p$
- bases (P, Q) and (R, S) of $E[2^n]$ and $E[3^m]$ (recall $E[k] \cong \mathbb{Z}/k \times \mathbb{Z}/k$)

Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \ldots 2^n - 1\}$		$b \xleftarrow{\text{random}} \{0 \ldots 3^m - 1\}$
$A := \langle P + [a]Q \rangle$		$B := \langle R + [b]S \rangle$
compute $\varphi_A : E \to E/A$		compute $\varphi_B : E \to E/B$
$E/A,\ \varphi_A(R),\ \varphi_A(S)$		$E/B,\ \varphi_B(P),\ \varphi_B(Q)$
$A' := \langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$		$B' := \langle \varphi_A(R) + [b]\varphi_A(S) \rangle$
$s := j\big((E/B)/A'\big)$		$s := j\big((E/A)/B'\big)$

Decomposing smooth isogenies

- In SIDH, $\#A = 2^n$ and $\#B = 3^m$ are 'crypto-sized'.
  Vélu's formulas take $\Theta(\#G)$ to compute $\varphi_G \colon E \to E/G$.

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!! Evaluate φ_G as a chain of small-degree isogenies:
For $G \cong \mathbb{Z}/\ell^k$, set $\ker \psi_i := [\ell^{k-i}](\psi_{i-1} \circ \cdots \circ \psi_1)(G)$.

$$\begin{tikzcd} E \arrow[r, "\psi_1"] \arrow[rrrr, bend right, "\varphi_G"'] & E_1 \arrow[r, "\psi_2"] & \ldots \arrow[r, "\psi_{k-1}"] & E_{k-1} \arrow[r, "\psi_k"] & E/G \end{tikzcd}$$

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$\rightsquigarrow$ Complexity: $O(k^2 \cdot \ell)$. Exponentially smaller than ℓ^k!
'Optimal strategy' improves this to $O(k \log k \cdot \ell)$.

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- BTW: The choice of p makes sure everything stays over $\mathbb{F}_{p^2}$.

Security of SIDH

The SIDH graph has size $\lfloor p/12 \rfloor + \varepsilon$.
Each secret isogeny φ_A, φ_B is a walk of about $\log p/2$ steps.
(Alice & Bob can choose from about $\sqrt{p}$ secret keys each.)

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Classical attacks:

- Cannot reuse keys without extra caution. (next slide)
- Meet-in-the-middle: $\tilde{\mathcal{O}}(p^{1/4})$ time & space.
- Collision finding: $\tilde{\mathcal{O}}(p^{3/8}/\sqrt{memory}/cores)$.

Security of SIDH

The SIDH graph has size $\lfloor p/12 \rfloor + \varepsilon$.
Each secret isogeny φ_A, φ_B is a walk of about $\log p/2$ steps.
(Alice & Bob can choose from about $\sqrt{p}$ secret keys each.)

Classical attacks:

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Quantum attacks:

- Claw finding: claimed $\tilde{\mathcal{O}}(p^{1/6})$. Newer paper says $\tilde{\mathcal{O}}(p^{1/4})$:

  "An adversary with enough quantum memory to run Tani's algorithm with the query-optimal parameters could break SIKE faster by using the classical control hardware to run van Oorschot–Wiener."

Thou shalt not reuse SIDH keys

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Validating that Bob is honest is $\approx$ as hard as breaking SIDH.

$\Longrightarrow$ only usable with ephemeral keys or as a KEM '*SIKE*'.


Questions?