Introduction to isogeny-based cryptography

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AIM, San Jose, 7 February 2019

Words are hard

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>
> — a lattice-based crypto researcher

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Mnemonic:

“I so genius!”

Diffie–Hellman key exchange ’76

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today elliptic curves)
- an element $g \in G$ of prime order q

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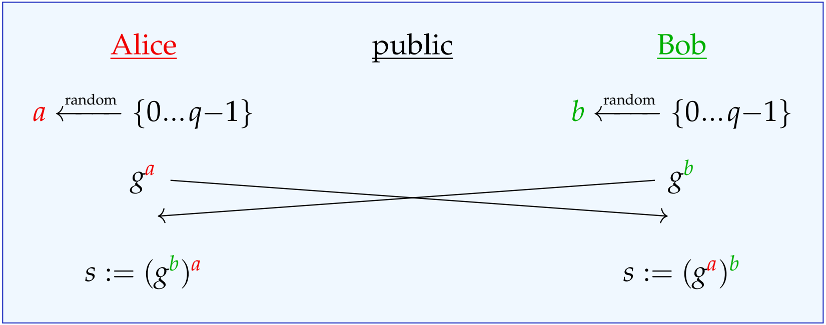

Alice

public

Bob

$a \xleftarrow{\text{random}} \{0\ldots q-1\}$

$b \xleftarrow{\text{random}} \{0\ldots q-1\}$

g^a

g^b

$s := (g^b)^a$

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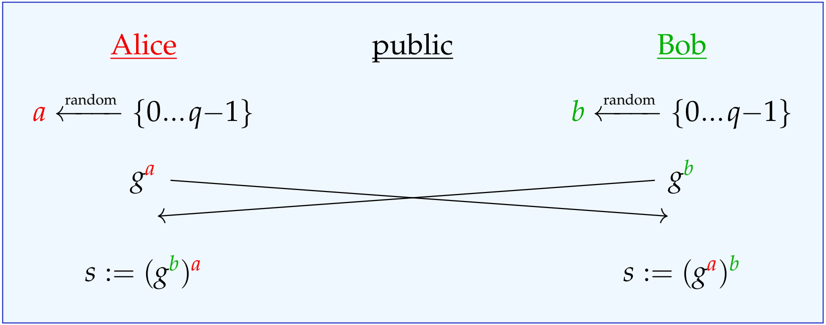

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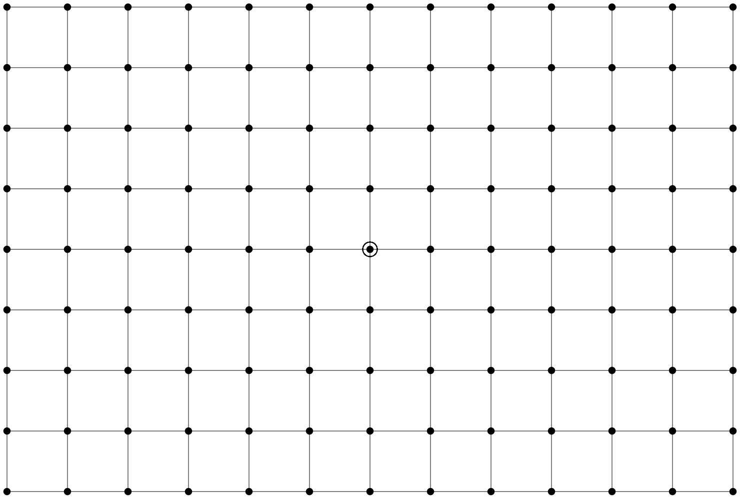
Fundamental reason this works: $\cdot^a$ and $\cdot^b$ are commutative!

Graph walking Diffie–Hellman?



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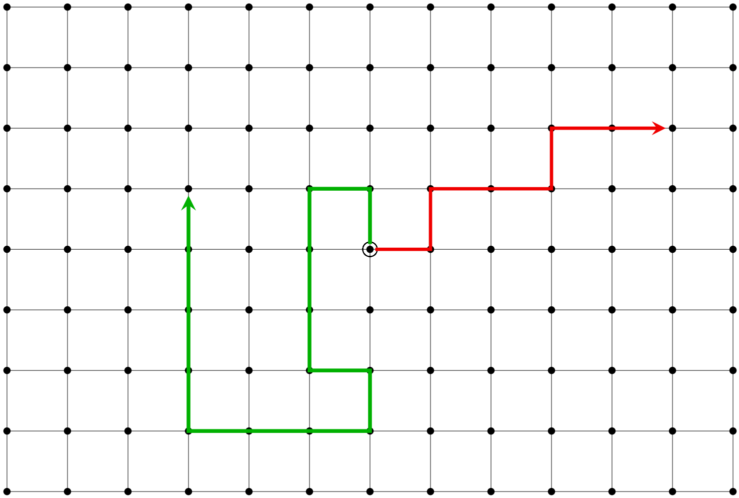

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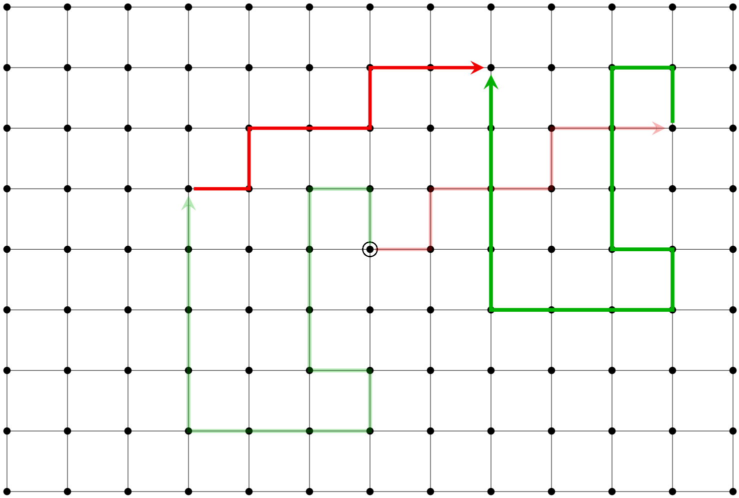

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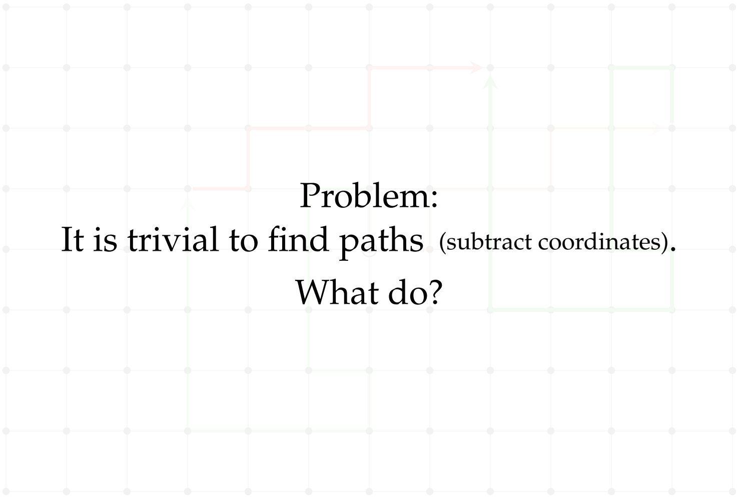

Problem:

It is trivial to find paths (subtract coordinates).

What do?

Big picture

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  That is: some *well-behaved* ‘directions’ to describe paths. More later.

It is easy to construct graphs that satisfy *almost* all of these —
not enough for crypto!

There are several more-or-less equivalent viewpoints.
I will focus on one of them, hence omit many *fun* details.
Please ask me about stuff!

Stand back!

We're going to do math.

(worry not: only 4 ~~tough~~ exciting slides ahead!)

Math slide #1: Elliptic curves *(nodes)*

An elliptic curve (modulo details) is given by an equation

$$E\colon\ y^2 = x^3 + ax + b.$$

A point on E is a solution to this equation *or* the ‘fake’ point ∞.

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E is an abelian group: we can ‘add’ points.

- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

$$\left(\lambda^2 - x_1 - x_2,\; \lambda(2x_1 + x_2 - \lambda^2) - y_1\right)$$

where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + a}{2y_1}$ otherwise.

do **not** remember these formulas!

Math slide #2: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$

- given by rational functions
- that is a group homomorphism.

The degree of a separable* isogeny is the size of its kernel.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m]\colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota\colon (x,y) \mapsto (-x, \sqrt{-1}\cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}\,.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$ defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

Math slide #3: Fields of definition

Until now: Everything over the algebraic closure.
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An elliptic curve/point/isogeny is defined over k
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Until now: Everything over the algebraic closure.
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An elliptic curve/point/isogeny is defined over k
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For E defined over k, let $E(k)$ be the points of E defined over k.

Math slide #4: Supersingular isogeny graphs

Let p be a prime, q a power of p, and ℓ a positive integer $\notin p\mathbb{Z}$.

An elliptic curve $E/\mathbb{F}_q$ is *supersingular* if $p \mid q + 1 - \#E(\mathbb{F}_q)$.

We care about the cases $\#E(\mathbb{F}_p) = p + 1$ and $\#E(\mathbb{F}_{p^2}) = (p + 1)^2$.

$\rightsquigarrow$ easy way to control the group structure by choosing p!

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Let $S \not\ni p$ denote a set of positive, pairwise coprime integers.

The supersingular S-isogeny graph over $\mathbb{F}_q$ consists of...

- isomorphism classes of supersingular elliptic curves
- with equivalence classes[1] of ℓ-isogenies ($\ell \in S$) as edges;

both defined over $\mathbb{F}_q$.

[1] Two isogenies $\varphi \colon E \to E'$ and $\psi \colon E \to E''$ are identified if $\psi = \iota \circ \varphi$ for some isomorphism $\iota \colon E' \to E''$.

The beauty and the beast

Components of the isogeny graphs look as follows:

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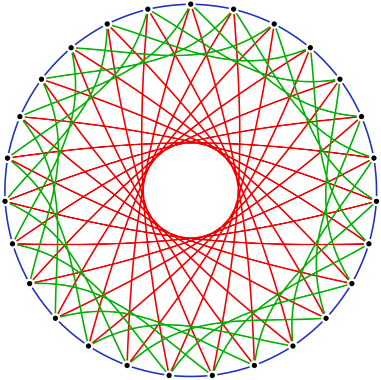

$S = \{3, 5, 7\}$, $q = 419$

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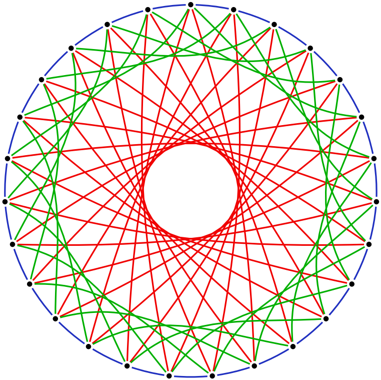

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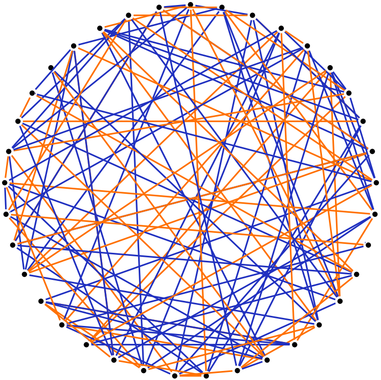
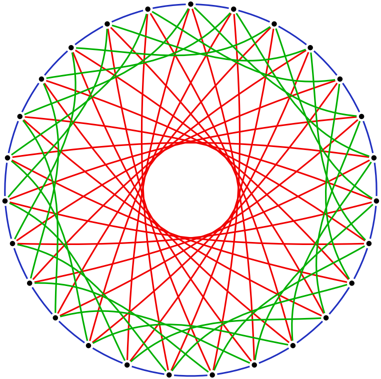


$S = \{2, 3\}$, $q = 431^2$

The beauty and the beast

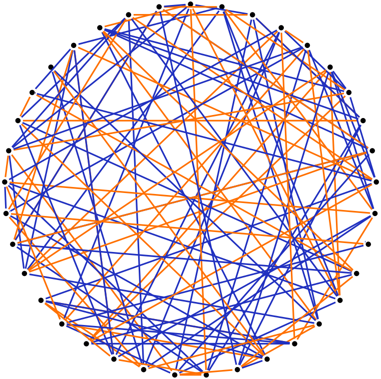
At this time, there are two distinct families of systems:



$q = p$

CSIDH [ˈsiːˌsaɪd]

`https://csidh.isogeny.org`



$q = p^2$

SIDH

`https://sike.org`

[ˈsiːˌsaɪd]

CSIDH

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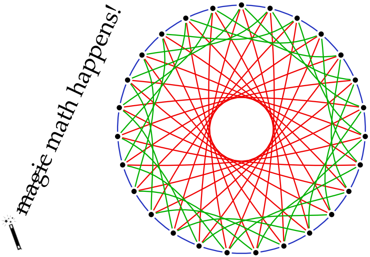

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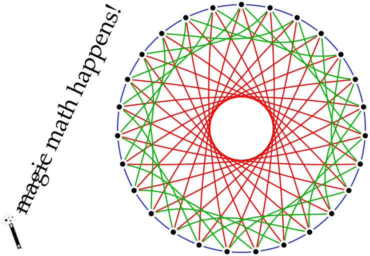
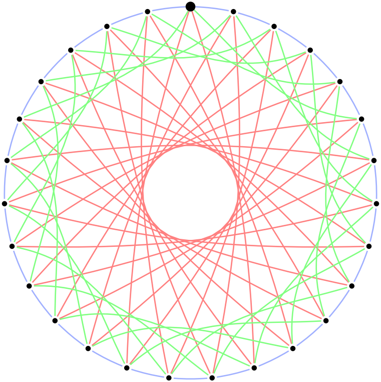


- Walking 'left' and 'right' on any ℓ_i-subgraph is efficient.

CSIDH key exchange

Alice

$[+, +, -, -]$



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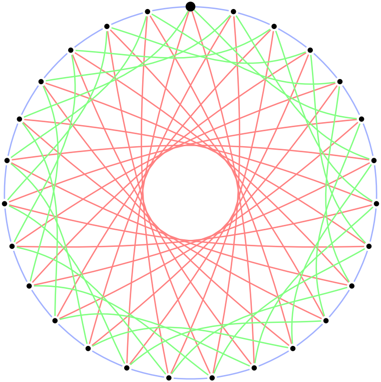

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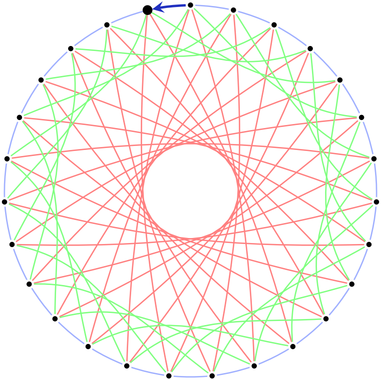

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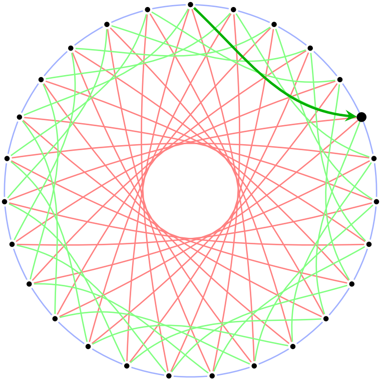

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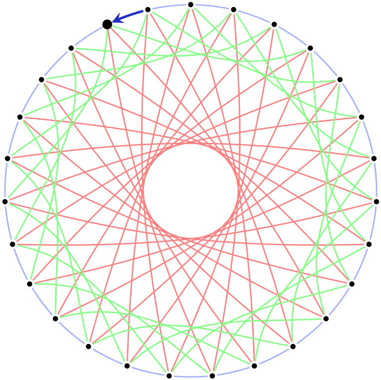

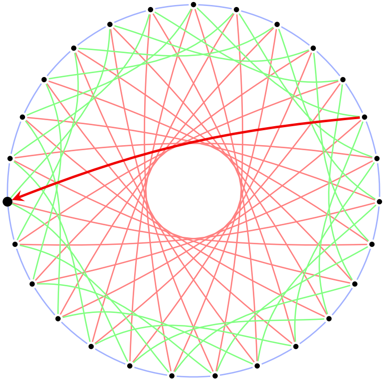

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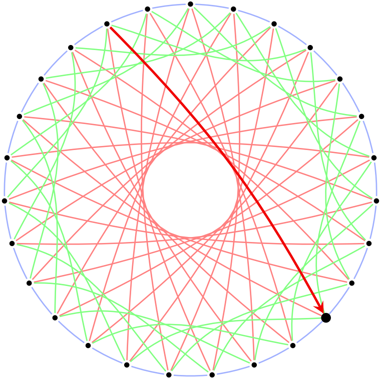

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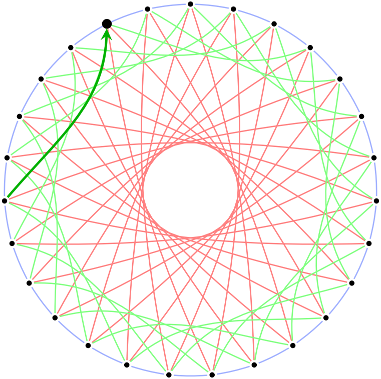

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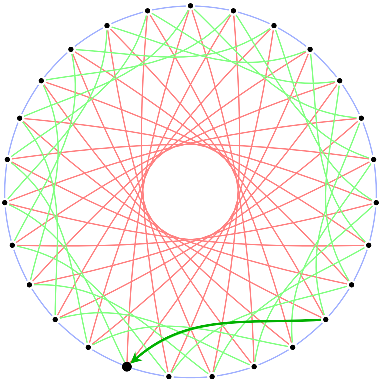

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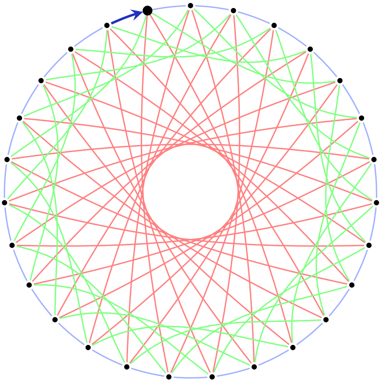

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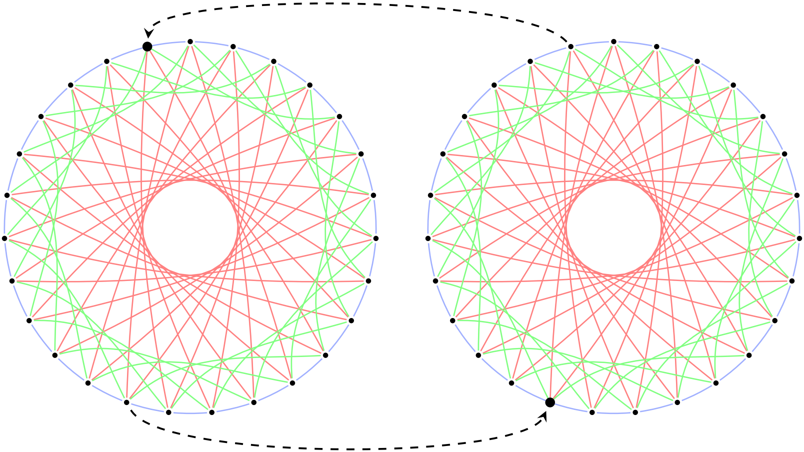

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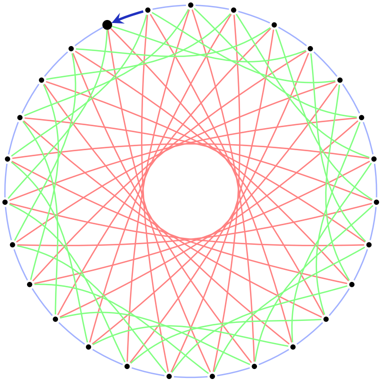

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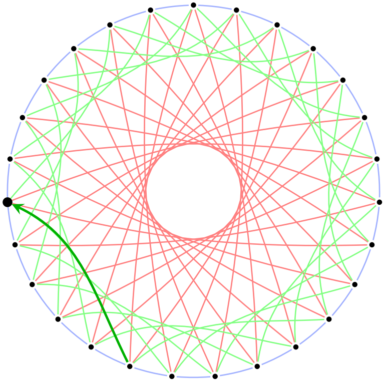

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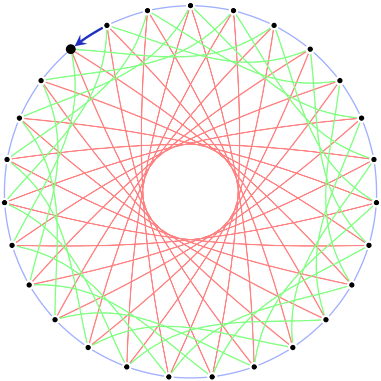

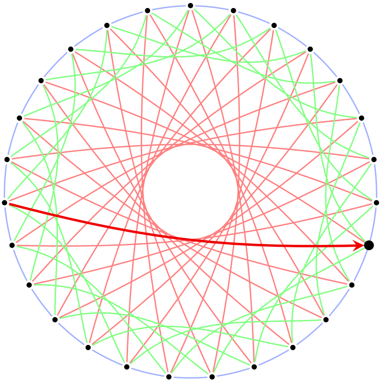

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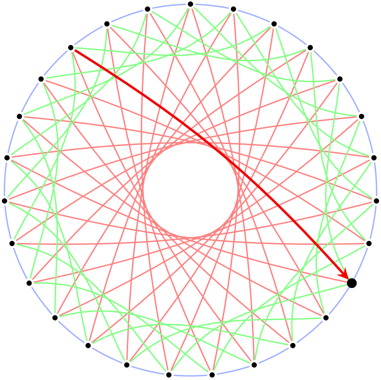

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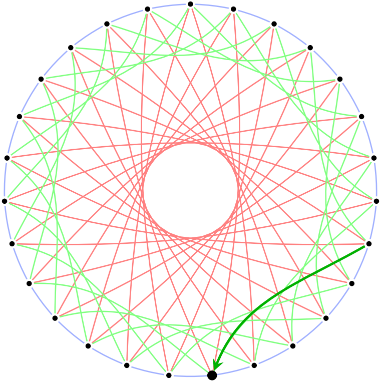

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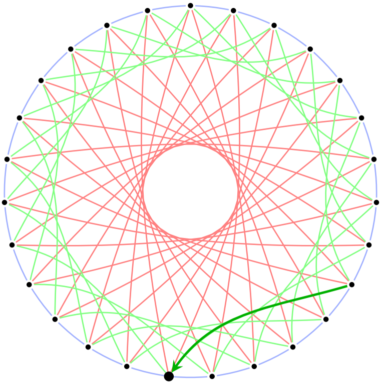

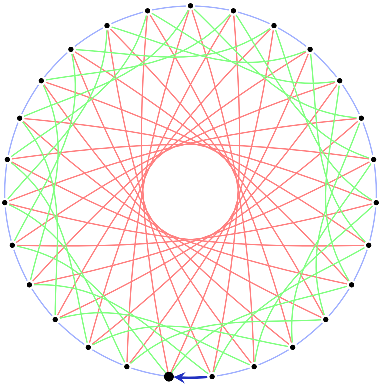

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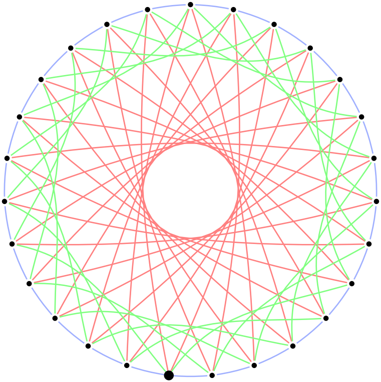

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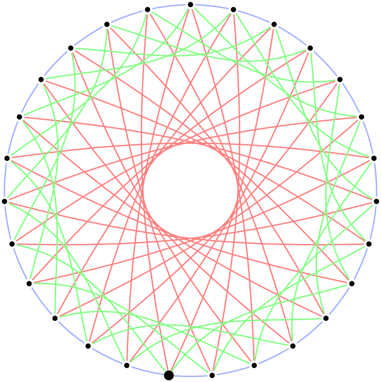

Has anyone seen my class group action?

Cycles are compatible: [right then left] = [left then right]
$\rightsquigarrow$ only need to keep track of total step counts for each ℓ_i.

Example: $[+,+,-,-,-,+,-,-]$ just becomes $(+1,\quad 0,-3) \in \mathbb{Z}^3$.

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There is a group action of $(\mathbb{Z}^n, +)$ on our set of curves X!

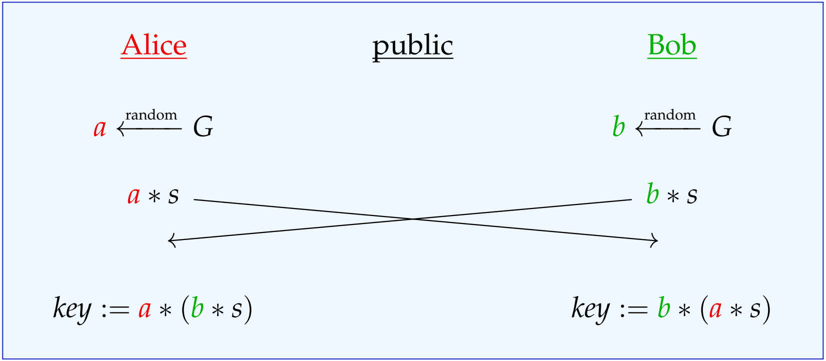
This action is transitive (for big enough n), but not free.
*Obviously**, quotienting out vectors which act trivially yields a group isomorphic to the ideal-class group $\mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$.

(This is because the curves in X have $\mathbb{F}_p$-endomorphism ring $\mathbb{Z}[\pi] \cong \mathbb{Z}[\sqrt{-p}]$. A prime ideal in $\mathbb{Z}[\pi]$ of norm ℓ corresponds to one of two eigenspaces of the Frobenius endomorphism π on the ℓ-torsion, which correspond to horizontal ℓ-isogenies that preserve the endomorphism ring.)

Cryptographic group actions

Previous slide: Free, transitive group action of $\mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$ on X.

Like in the CSIDH example before, we *generally* get a DH-like key exchange from a group action $G \times S \to S$:



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$a \xleftarrow{\text{random}} G$

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$a * s$

$b * s$

$key := a * (b * s)$

$key := b * (a * s)$

Why no Shor?

Shor computes α from $h = g^{\alpha}$ by finding the kernel of the map

$$f\colon\ \mathbb{Z}^2 \to G,\ (x, y) \mapsto g^{x} \underset{\uparrow}{\cdot} h^{y}$$

For general group actions, we cannot compose $a * s$ and $b * s$!

Security of CSIDH

Core problem:
Given $E, E' \in X$, find a smooth-degree isogeny $E \to E'$.
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The size of X is $\#\mathrm{cl}(\mathbb{Z}[\sqrt{-p}]) \approx \sqrt{p}$.

$\rightsquigarrow$ best known classical attack: meet-in-the-middle, $\tilde{\mathcal{O}}(p^{1/4})$.

Security of CSIDH

Core problem:
Given $E, E' \in X$, find a smooth-degree isogeny $E \to E'$.
Given $E, E' \in X$, find a smooth ideal $\mathfrak{a}$ of $\mathbb{Z}[\sqrt{-p}]$ with $[\mathfrak{a}]E = E'$.

The size of X is $\#\mathrm{cl}(\mathbb{Z}[\sqrt{-p}]) \approx \sqrt{p}$.

$\rightsquigarrow$ best known classical attack: meet-in-the-middle, $\tilde{\mathcal{O}}(p^{1/4})$.

Solving abelian hidden shift breaks CSIDH.

$\rightsquigarrow$ quantum subexponential attack (Kuperberg's algorithm).

Can we avoid Kuperberg's algorithm?

With great commutative group action
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- Problem: also no more intrinsic sense of direction.

  "It all bloody looks the same!" — a famous isogeny cryptographer

$\rightsquigarrow$ need extra information to let Alice&Bob's walks commute.

Math slide #5: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is called E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

[1] (up to isomorphism of E')

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Vélu operates in the field where the points in G live.
$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$
$\rightsquigarrow$ this is why we use supersingular curves!

[1] (up to isomorphism of E')

Now:
SIDH

(...whose name doesn't allow for nice pictures of beaches...)

Wikipedia about SIDH...

"While several steps of SIDH involve complex isogeny calculations, the overall flow of SIDH for parties A and B is straightforward for those familiar with a Diffie–Hellman key exchange or its elliptic curve variant. [...]

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Setup.

1. A prime of the form $p = w_A^{e_A} \cdot w_B^{e_B} \cdot f \pm 1$.
2. A supersingular elliptic curve E over $\mathbb{F}_{p^2}$.
3. Fixed elliptic points P_A, Q_A, P_B, Q_B on E.
4. The order of P_A and Q_A is $(w_A)^{e_A}$.
5. The order of P_B and Q_B is $(w_B)^{e_B}$.

Key exchange. [...]

1A. A generates two random integers $m_A, n_A < (w_A)^{e_A}$.
2A. A generates $R_A := m_A \cdot (P_A) + n_A \cdot (Q_A)$.
3A. A uses the point R_A to create an isogeny mapping $\phi_A : E \to E_A$ and curve E_A isogenous to E.
4A. A applies ϕ_A to P_B and Q_B to form two points on E_A : $\phi_A(P_B)$ and $\phi_A(Q_B)$.
5A. A sends to B E_A, $\phi_A(P_B)$, and $\phi_A(Q_B)$.
1B–4B. Same as A1 through A4, but with A and B subscripts swapped.
5B. B sends to A E_B, $\phi_B(P_A)$, and $\phi_B(Q_A)$.
6A. A has $m_A, n_A, \phi_B(P_A)$, and $\phi_B(Q_A)$ and forms $S_{BA} := m_A(\phi_B(P_A)) + n_A(\phi_B(Q_A))$.
7A. A uses S_{BA} to create an isogeny mapping ψ_{BA}.
8A. A uses ψ_{BA} to create an elliptic curve E_{BA} which is isogenous to E.
9A. A computes $K :=$ j-invariant (j_{BA}) of the curve E_{BA}.
6B. Similarly, B has $m_B, n_B, \phi_A(P_B)$, and $\phi_A(Q_B)$ and forms $S_{AB} = m_B(\phi_A(P_B)) + n_B(\phi_A(Q_B))$.
7B. B uses S_{AB} to create an isogeny mapping ψ_{AB}.
8B. B uses ψ_{AB} to create an elliptic curve E_{AB} which is isogenous to Ek
9B. B computes $K :=$ j-invariant (j_{AB}) of the curve E_{AB}.

The curves E_{AB} and E_{BA} are guaranteed to have the same j-invariant.”

SIDH: High-level view

$$\begin{tikzcd}
E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E/A \arrow[d, "\varphi_{B'}"] \\
E/B \arrow[r, "\varphi_{A'}"'] & E/\langle A, B\rangle
\end{tikzcd}$$

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- Alice & Bob pick secret subgroups A and B of E.
- Alice computes $\varphi_A \colon E \to E/A$; Bob computes $\varphi_B \colon E \to E/B$.
  (These isogenies correspond to walking on the isogeny graph.)

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- Alice and Bob transmit the values E/A and E/B.

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- Alice and Bob transmit the values E/A and E/B.
- Alice somehow obtains $A' := \varphi_B(A)$. (Similar for Bob.)
- They both compute the shared secret

  $$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

SIDH's auxiliary points

Previous slide: "Alice somehow obtains $A' := \varphi_B(A)$."

Alice knows only A, Bob knows only φ_B. Hm.

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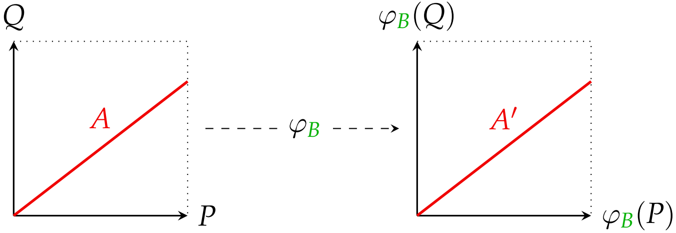
Alice knows only A, Bob knows only φ_B. Hm.

Solution: φ_B is a group homomorphism!

- Alice picks A as $\langle P + [a]Q \rangle$ for fixed public $P, Q \in E$.
- Bob includes $\varphi_B(P)$ and $\varphi_B(Q)$ in his public key.

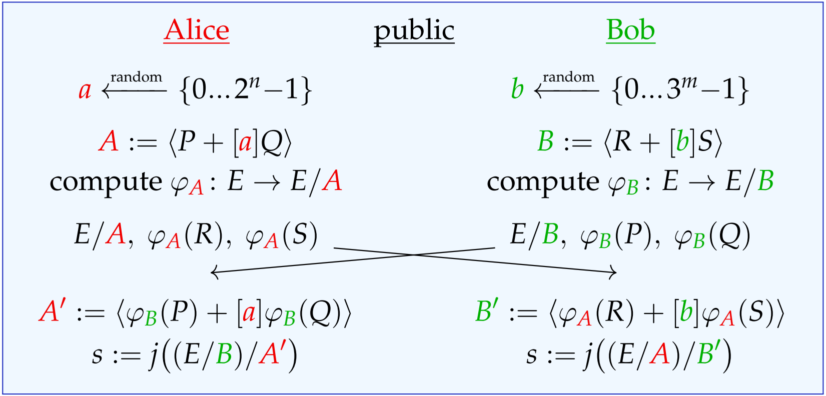
$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$!



SIDH in one slide

Public parameters:

- a large prime $p = 2^n 3^m - 1$ and a supersingular $E/\mathbb{F}_p$
- bases (P, Q) and (R, S) of $E[2^n]$ and $E[3^m]$



Alice

public

Bob

$a \xleftarrow{\text{random}} \{0 \ldots 2^n - 1\}$

$b \xleftarrow{\text{random}} \{0 \ldots 3^m - 1\}$

$A := \langle P + [a]Q \rangle$

compute $\varphi_A : E \to E/A$

$B := \langle R + [b]S \rangle$

compute $\varphi_B : E \to E/B$

$E/A,\ \varphi_A(R),\ \varphi_A(S)$

$E/B,\ \varphi_B(P),\ \varphi_B(Q)$

$A' := \langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$

$s := j\big((E/B)/A'\big)$

$B' := \langle \varphi_A(R) + [b]\varphi_A(S) \rangle$

$s := j\big((E/A)/B'\big)$

Security of SIDH

The SIDH graph has size $\lfloor p/12 \rfloor + \varepsilon$.
Each secret isogeny φ_A, φ_B is a walk of about $\log p/2$ steps.
(Alice & Bob can choose from about $\sqrt{p}$ secret keys each.)

[1] `https://ia.cr/2019/103`

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Classical attacks:

- Cannot reuse keys without extra caution.
- Meet-in-the-middle: $\tilde{\mathcal{O}}(p^{1/4})$ time & space.
- Collision finding: $\tilde{\mathcal{O}}(p^{3/8}/\sqrt{memory}/cores)$.

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Quantum attacks:

- Claw finding: claimed $\tilde{\mathcal{O}}(p^{1/6})$. New paper[1] says $\tilde{\mathcal{O}}(p^{1/4})$:

  "An adversary with enough quantum memory to run Tani's algorithm with the query-optimal parameters could break SIKE faster by using the classical control hardware to run van Oorschot–Wiener."

[1] https://ia.cr/2019/103

Open and half-open questions

CSIDH:

How costly is breaking CSIDH with Kuperberg's algorithm?

Is Kuperberg's algorithm optimal for abelian hidden shift?

Are there any non-generic quantum attacks?

SIDH:

Do the points $\varphi_B(P), \varphi_B(Q)$ reveal too much information?

Can we phrase SIDH as a hidden-subgroup problem?

Are there any non-generic quantum attacks?

Thank you!