Diffie–Hellman reductions

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Hardness reductions in cryptography

Recall RSA encryption (simplified special case):

- Private key: two big prime numbers p, q.
- Public key: their product $n = pq$.
- Encrypt: compute $c = m^{65537} \bmod n$.
- Decrypt: compute $m = c^{65537^{-1} \bmod \operatorname{lcm}(p-1, q-1)} \bmod pq$.

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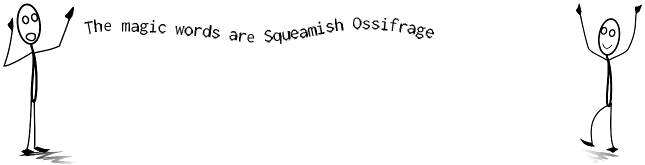
If yes:
No point attacking RSA specifically; just focus on factoring.

Diffie–Hellman key exchange (very big picture)



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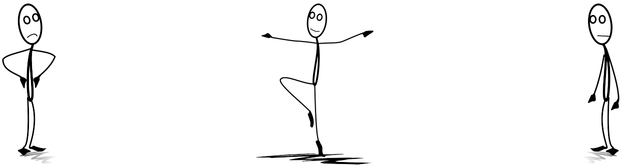

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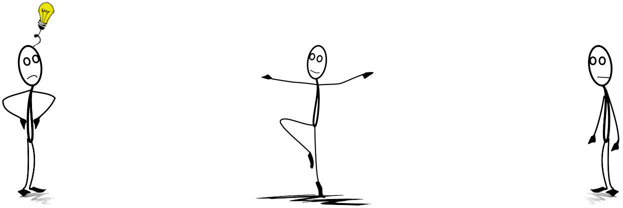

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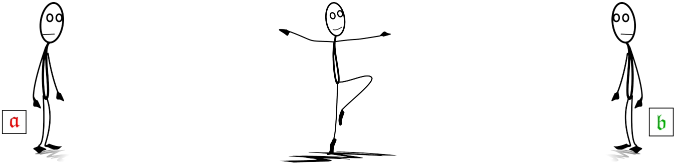

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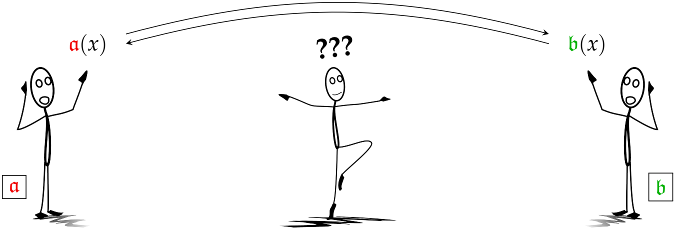

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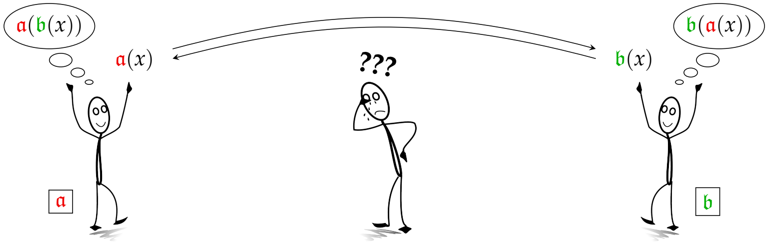

- Private keys:
  efficient functions $\mathfrak{a}, \mathfrak{b} \colon X \to X$ such that $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{b} \circ \mathfrak{a}$.
- Public keys: the elements $\mathfrak{a}(x), \mathfrak{b}(x) \in X$.
- Shared secret: the element $\mathfrak{a}(\mathfrak{b}(x)) = \mathfrak{b}(\mathfrak{a}(x))$.

This talk

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- Standard proof technique:
  Use a black-box 'oracle'

  $$\mathcal{O}\colon\ (x, \mathfrak{a}(x), \mathfrak{b}(x)) \longmapsto \mathfrak{a}(\mathfrak{b}(x))$$

  to construct an efficient algorithm

  $$\mathcal{A}(\mathcal{O})\colon\ \mathfrak{a}(x) \longmapsto \mathfrak{a}\,.$$

  (The oracle formalizes an attack that we don't know yet.)

Group-based Diffie–Hellman

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Examples:

- Multiplicative groups of finite fields $(\mathbb{F}_q^*, \cdot)$.
- Elliptic curves $E\colon y^2 = x^3 + Ax^2 + x$ with 'weird' addition.

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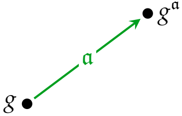
  Compute $\mathfrak{a}$ from $g, g^{\mathfrak{a}}$.



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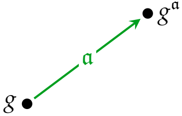
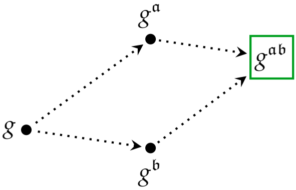


- Computational Diffie–Hellman problem (**CDH**)
  Compute $g^{\mathfrak{a}\mathfrak{b}}$ from $g, g^{\mathfrak{a}}, g^{\mathfrak{b}}$.



Generic complexity of DLP (Pohlig–Hellman 1978)

- Upshot: If the factorization of $|G|$ is $p_1^{e_1} \cdots p_r^{e_r}$, then one can solve DLP in $O(\sum_{i=1}^{r} e_i \cdot (\log |G| + \sqrt{p_i}))$ group operations.

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!! There are many groups where one can solve DLP faster.

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Anyone can...

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- exponentiate exponents: square-and-multiply using $\circlearrowright$.
- invert exponents: $g^{1/\mathfrak{a}} = g^{\mathfrak{a}^{\varphi(|G|)-1}}$ if $\gcd(\mathfrak{a}, |G|) = 1$ using $\circlearrowright$.

Black-box rings

- We interpret $g^{\mathfrak{a}}$ as labels for the hidden elements $\mathfrak{a}$.
- With a CDH oracle we can perform arbitrary ring operations $(+, -, \cdot, /)$ on these hidden representations.
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We mostly care about black-box fields:
For discrete logarithms, it's sufficient to consider prime-order g.

First result: den Boer (1988)

Let $G = \mathbb{F}_p^*$, write $R = \mathbb{Z}/|G| = \mathbb{Z}/(p-1)$,
and suppose $|R^*| = \varphi(p-1)$ is smooth.
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Proof:

- Suppose (for simplicity) that R^* is cyclic and find a generator $\mathfrak{g}$.
- Encode $\mathfrak{g}$ to a black-box element $\lceil \mathfrak{g} \rfloor$ of R.
- Solve the DLP $(\lceil \mathfrak{g} \rfloor, \lceil \mathfrak{a} \rfloor)$ in the hidden version of R^* using Pohlig–Hellman. We get $k \in \mathbb{Z}$ such that $g^{\mathfrak{a}} = g^{\mathfrak{g}^k}$.
- Simply compute $\mathfrak{a}$ as the power $\mathfrak{g}^k \in R^*$.

Observation:
There is nothing special about using R^* in the exponents;
in principle anything expressible as field operations works.

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Example: For $|G| = p$ with $p^d - 1$ smooth, we can use $\mathbb{F}_{p^d}^*$.

Second result: Maurer (1994)

Let G be of prime order p, write $R = \mathbb{F}_p$,
and suppose $E\colon y^2 = x^3 + Ax^2 + x \;/\mathbb{F}_p$ has smooth order.
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Proof:

- Find a generator point G on E.
- Hope that $\mathfrak{a}$ is an x-coordinate on the curve ($\Pr \approx 1/2$).
  Compute (black-box) the corresponding y-coordinate $\lceil \mathfrak{y} \rfloor$,
  giving a black-box elliptic-curve point $\lceil P \rfloor = (\lceil \mathfrak{a} \rfloor, \lceil \mathfrak{y} \rfloor)$.
  (If $\lceil \mathfrak{y} \rfloor^2 \neq \lceil \mathfrak{a} \rfloor^3 + \lceil A \rfloor \lceil \mathfrak{a} \rfloor^2 + \lceil \mathfrak{a} \rfloor$, then randomize $\lceil \mathfrak{a} \rfloor$ as $\lceil \mathfrak{a}' \rfloor = \lceil \mathfrak{a} \rfloor + \lceil \delta \rfloor$ and retry.)
- Solve the (black-box) DLP $(\lceil G \rfloor, \lceil P \rfloor)$ via Pohlig–Hellman.
  We get $k \in \mathbb{Z}$ such that $(\mathfrak{a}, \mathfrak{y}) = [k]G$.
- Simply compute $\mathfrak{a}$ as the x-coordinate of $[k]G$.

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Unknown in general, but likely.

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...and they lived happily ever after**??**

Shor's algorithm (1994)

Shor's algorithm breaks all group-based DH instantiations.

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a map with an unknown period lattice $\Lambda \subseteq \mathbb{Z}^n$, such that

$$f(v+\tau) = f(v)$$

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Given such f and some size constraints on Λ,
Shor's algorithm recovers a basis of Λ in polynomial time.

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Application:

For a DLP instance $(g, h = g^k)$ in a cyclic group G of order q, the (publicly computable) function

$$\begin{aligned} f\colon\ \mathbb{Z}^2 &\longrightarrow G \\ (x, y) &\longmapsto g^x \cdot h^y \end{aligned}$$

has period $\Lambda = \langle (k, -1), (q, 0) \rangle \subseteq \mathbb{Z}^2$, which Shor can recover.

And now...
For something totally different.

Diffie–Hellman from group actions (2006)

Let G be a group, X a set. A group action of G on X is a map

$$*\colon\ G \times X \longrightarrow X$$

such that $\mathrm{id} * x = x$ and $(\mathfrak{g} \cdot \mathfrak{h}) * x = \mathfrak{g} * (\mathfrak{h} * x)$.

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This suggests an evident Diffie–Hellman scheme:
Let G be finite and commutative and fix $x \in X$.

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Example: CSIDH [ˈsiːˌsaɪd] (2018) [joint w/ Castryck, Lange, Martindale, Renes]

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- exponentiate: square-and-multiply using $\circlearrowright$.
- invert: $\lceil \mathfrak{a}^{-1} \rfloor = \lceil \mathfrak{a} \rfloor^{|G|-1}$ using $\circlearrowright$.

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Theorem. There is a polynomial-time <u>quantum</u> equivalence between the CDH and DLP problems <u>for group actions</u>.

Proof:

- Compute a set of generators $\mathfrak{g}_1, \ldots, \mathfrak{g}_r \in G$.
- Apply Shor's algorithm to the map

$$\begin{aligned} f\colon \quad \mathbb{Z}^r \times \mathbb{Z} &\longrightarrow X \\ (x_1, \ldots, x_r, y) &\longmapsto (\mathfrak{g}_1^{x_1} \cdots \mathfrak{g}_r^{x_r}) * \lceil \mathfrak{a} \rfloor^y . \end{aligned}$$

- Any period vector of the form $(x_1, \ldots, x_r, 1)$ yields the desired element $\mathfrak{a} = \mathfrak{g}_1^{-x_1} \cdots \mathfrak{g}_r^{-x_r}$.

An open question

- Can we get similar results if the CDH oracle $(x, \mathfrak{a} * x, \mathfrak{b} * x) \mapsto \mathfrak{ab} * x$ is unreliable?

  Classical case: Yes, by repeatedly blinding the inputs, unblinding the outputs, and using majority vote.

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Thank you!