Diffie–Hellman reductions

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Hardness reductions in cryptography

Recall RSA encryption (simplified special case):

- Private key: two big prime numbers p, q.
- Public key: their product $n = pq$.
- Encrypt: compute $c = m^{65537} \bmod n$.
- Decrypt: compute $m = c^{65537^{-1} \bmod \operatorname{lcm}(p-1, q-1)} \bmod pq$.

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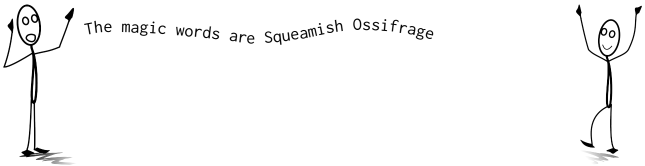
If yes:
No point attacking RSA specifically; just focus on factoring.

Diffie–Hellman key exchange (very big picture)



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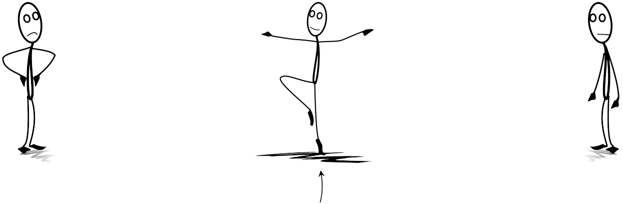

evil eavesdropper Eve!

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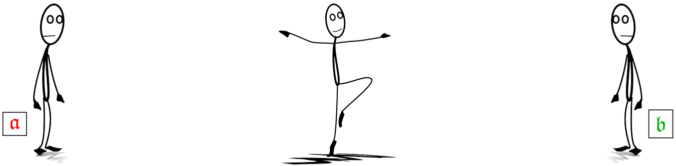

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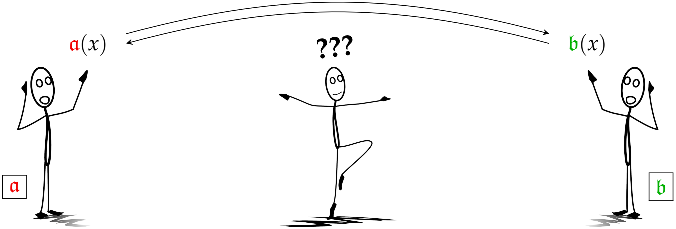

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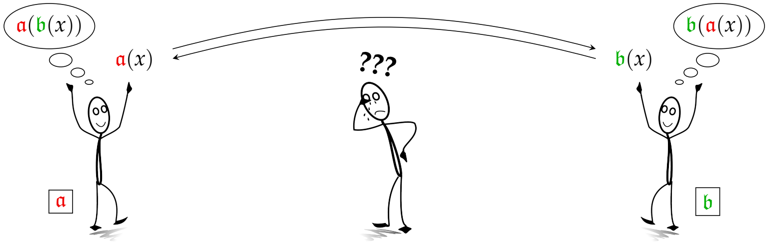

- <u>Private</u> keys:
  efficient functions $\mathfrak{a}, \mathfrak{b} \colon X \to X$ such that $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{b} \circ \mathfrak{a}$.
- <u>Public</u> keys: the elements $\mathfrak{a}(x), \mathfrak{b}(x) \in X$.
- <u>Shared</u> secret: the element $\mathfrak{a}(\mathfrak{b}(x)) = \mathfrak{b}(\mathfrak{a}(x))$.

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Intuition: *Every* efficient $\mathcal{O}$ leads to an efficient $\mathcal{A}(\mathcal{O})$.

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Examples:

- Multiplicative groups of finite fields $(\mathbb{F}_q^*, \cdot)$.
- Elliptic curves $E\colon y^2 = x^3 + Ax^2 + x$ with ‘weird’ addition.

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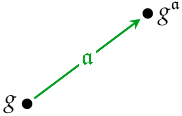
  Compute $\mathfrak{a}$ from $g, g^{\mathfrak{a}}$.



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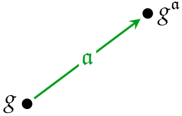
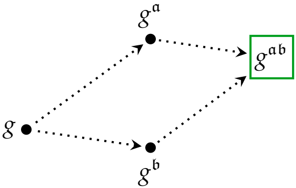


- <u>Computational Diffie–Hellman problem (**CDH**)</u>
  Compute $g^{\mathfrak{a}\mathfrak{b}}$ from $g, g^{\mathfrak{a}}, g^{\mathfrak{b}}$.



Generic complexity of DLP (Pohlig–Hellman 1978)

- Upshot: If the factorization of $|G|$ is $p_1^{e_1} \cdots p_r^{e_r}$, then one can solve DLP in $O\big(\sum_{i=1}^{r} e_i \cdot (\log|G| + \sqrt{p_i})\big)$ group operations.

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!! There are many groups where one can solve DLP faster.

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- invert exponents: $g^{1/\mathfrak{a}} = g^{\mathfrak{a}^{\varphi(|G|)-1}}$ if $\gcd(\mathfrak{a}, |G|) = 1$ using $\circlearrowleft$.

Black-box rings

- We interpret $g^{\mathfrak{a}}$ as labels for the hidden elements $\mathfrak{a}$.
- With a CDH oracle we can perform arbitrary ring operations $(+,-,\cdot,/)$ on these hidden representations.
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We mostly care about black-box fields:
For discrete logarithms, it's sufficient to consider prime-order g.

First result: den Boer (1988)

Let $G = \mathbb{F}_p^*$, write $R = \mathbb{Z}/|G| = \mathbb{Z}/(p-1)$,
and suppose $|R^*| = \varphi(p-1)$ is smooth.
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- Simply compute $\mathfrak{a}$ as the power $\mathfrak{h}^k \in R^*$.

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There is nothing special about using R^* in the exponents;
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Auxiliary groups

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in principle anything expressible as field operations works.

This is known as an auxiliary group:
A smooth-order algebraic group over the black-box field.

Slightly late 'about me' slide

I played too many hacking competitions in $[2013; +\infty)$.

Example auxiliary group: $\mathbb{F}_{p^2}^*$

A challenge I made for a CTF last year:

Solve a DLP $(g, g^{\mathfrak{a}})$ in a black-box group of order $p = 2^{48} - 5297$ using at most 2^{14} queries to `mul`, `inv`, and `dhp`.

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```
p = 2**48 - 5297            # |G| = p
mul = lambda x,y: (x+y)%p   # g^x * g^y = g^(x+y)
inv = lambda x: (-x)%p      # (g^x)^-1  = g^(-x)
dhp = lambda x,y: x*y%p     # enjoy your oracle!

aes = AES.new(os.urandom(16), AES.MODE_ECB)
enc = lambda x: aes.encrypt(x.to_bytes(16, 'big')).hex()
dec = lambda y: int.from_bytes(aes.decrypt(bytes.fromhex(y)), 'big')

g, a = 1, random.randrange(p)
print(enc(g), enc(y))

for _ in range(2**14):
    q = input().strip().split()
    if q[0] == 'mul': print(enc(mul(dec(q[1]), dec(q[2]))))
    if q[0] == 'inv': print(enc(inv(dec(q[1]))))
    if q[0] == 'dhp': print(enc(dhp(dec(q[1]), dec(q[2]))))

if int(input()) % p == a: print(open('flag.txt').read())
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- Observe that $\mathfrak{h}^{p+1} \in \mathbb{F}_p^*$, so $\mathfrak{h}^k = c\mathfrak{a} + c\mathrm{i}$ for some $c \in \mathbb{F}_p^*$. $\implies$ Divide $\mathfrak{h}^k$ by its i-coefficient to obtain $\mathfrak{a} + \mathrm{i}$, hence $\mathfrak{a}$!

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(This uses only $\sim$4500 queries. Intended solution $\sim$3 times as many.)

Second result: Maurer (1994)

Let G be of prime order p, write $R = \mathbb{F}_p$,
and suppose $E\colon y^2 = x^3 + Ax^2 + x \ /\mathbb{F}_p$ has smooth order.
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Then CDH is polynomial-time equivalent to DLP in G.

Proof:

- Find a generator point G on E.
- Hope that $\mathfrak{a}$ is an x-coordinate on the curve ($\Pr \approx 1/2$).
  Compute (black-box) the corresponding y-coordinate $\lceil \mathfrak{y} \rfloor$,
  giving a black-box elliptic-curve point $\lceil P \rfloor = (\lceil \mathfrak{a} \rfloor, \lceil \mathfrak{y} \rfloor)$.
  (If $\lceil \mathfrak{y} \rfloor^2 \neq \lceil \mathfrak{a} \rfloor^3 + \lceil A \rfloor \lceil \mathfrak{a} \rfloor^2 + \lceil \mathfrak{a} \rfloor$, then randomize $\lceil \mathfrak{a} \rfloor$ as $\lceil \mathfrak{a}' \rfloor = \lceil \mathfrak{a} \rfloor + \lceil \delta \rfloor$ and retry.)
- Solve the (black-box) DLP $(\lceil G \rfloor, \lceil P \rfloor)$ via Pohlig–Hellman.
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- Simply compute $\mathfrak{a}$ as the x-coordinate of $[k]G$.

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Let G be of prime order p, write $R = \mathbb{F}_p$,
and suppose $E\colon y^2 = x^3 + Ax^2 + x \ /\mathbb{F}_p$ has smooth order.
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Unknown in general, but likely.

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...and they lived happily ever after**??**

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a map with an unknown period lattice $\Lambda \subseteq \mathbb{Z}^n$, such that

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Given such f and some size constraints on Λ,
Shor's algorithm recovers a basis of Λ in polynomial time.

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Application:

For a DLP instance $(g, h = g^{\mathfrak{a}})$ in a cyclic group G of order q, the (publicly computable) function

$$\begin{aligned} f\colon\ \mathbb{Z}^2 &\longrightarrow G \\ (x, y) &\longmapsto g^x \cdot h^y \end{aligned}$$

has period $\Lambda = \langle (\mathfrak{a}, -1), (q, 0) \rangle \subseteq \mathbb{Z}^2$, which Shor can recover.

And now...
For something totally different.

Diffie–Hellman from group actions (2006)

Let G be a group, X a set. A group action of G on X is a map

$$*\colon\; G \times X \longrightarrow X$$

such that $\mathrm{id} * x = x$ and $(\mathfrak{g} \cdot \mathfrak{h}) * x = \mathfrak{g} * (\mathfrak{h} * x)$.

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This suggests an obvious Diffie–Hellman scheme:
Let G be finite and commutative and fix $x \in X$.

- Private keys: group elements $\mathfrak{a}, \mathfrak{b} \in G$.
- Public keys: the elements $\mathfrak{a} * x$, $\mathfrak{b} * x \in X$.
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Example: CSIDH [ˈsiːˌsaɪd] (2018) [joint w/ Castryck, Lange, Martindale, Renes]

The implicit group

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Theorem. There is a polynomial-time quantum equivalence between the CDH and DLP problems for group actions.

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Proof:

- Compute a set of generators $\mathfrak{g}_1, \ldots, \mathfrak{g}_r \in G$.
- Apply Shor's algorithm to the map

$$\begin{aligned} f\colon \quad \mathbb{Z}^r \times \mathbb{Z} &\longrightarrow X \\ (x_1, \ldots, x_r, y) &\longmapsto (\mathfrak{g}_1^{x_1} \cdots \mathfrak{g}_r^{x_r}) * \lceil \mathfrak{a} \rfloor^y . \end{aligned}$$

- Any period vector of the form $(x_1, \ldots, x_r, 1)$ yields the desired element $\mathfrak{a} = \mathfrak{g}_1^{-x_1} \cdots \mathfrak{g}_r^{-x_r}$.

Work in "progress"

- Can we get similar results in the group-action setting if the CDH oracle $(x, \mathfrak{a} * x, \mathfrak{b} * x) \mapsto \mathfrak{a}\mathfrak{b} * x$ is unreliable?

  Classical case: Yes, by repeatedly blinding the inputs, unblinding the outputs, and using majority vote.

  Here: Exponentially many queries in superposition; do we need *all* of them to be correct?

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And it might turn out to be impossible.

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Thank you!