Diffie–Hellman reductions

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Recall RSA encryption (simplified special case):

- **Private key:** two big prime numbers \( p, q \).
- **Public key:** their product \( n = pq \).
- **Encrypt:** compute \( c = m^{65537} \mod n \).
- **Decrypt:** compute \( m = c^{65537^{-1}} \mod \text{lcm}(p-1, q-1) \mod pq \).

Clearly, anyone who can factor \( n \) can decrypt.

Q: Can everyone capable of decrypting also factor \( n \)?

If yes: No point attacking RSA specifically; just focus on factoring.
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    a(b(x)) & = b(a(x)) \\
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Private keys: efficient functions $a, b: X \rightarrow X$ such that $a \circ b = b \circ a$.

Public keys: the elements $a(x), b(x) \in X$.

Shared secret: the element $a(b(x)) = b(a(x))$.

evil eavesdropper Eve!
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![Diagram of the Diffie–Hellman key exchange](image)
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This talk

Is computing $a(b(x))$ as hard as recovering $a$ or $b$?

Standard proof technique: Use a black-box 'oracle' $O: (x, a(x), b(x)) \mapsto \rightarrow a(b(x))$ to construct an efficient algorithm $A(O): a(x) \mapsto \rightarrow a$. Intuition: Every efficient $O$ leads to an efficient $A(O)$. 

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  **Intuition:** *Every* efficient $O$ leads to an efficient $A(O)$. 
Group-based Diffie–Hellman

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\( (G, \cdot) \) a finite group; \( a, b \) exponentiations.

- Private keys: \( a, b \in \mathbb{Z}/\text{ord } g \).
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Examples:
- Multiplicative groups of finite fields \((\mathbb{F}_q^*, \cdot)\).
- Elliptic curves \(E: y^2 = x^3 + Ax^2 + x\) with ‘weird’ addition.
Problems from Diffie–Hellman

Discrete-logarithm problem (DLP)
Compute $a$ from $g$, $g^a$.

Computational Diffie–Hellman problem (CDH)
Compute $g^{ab}$ from $g$, $g^a$, $g^b$. 

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  Compute $g^{ab}$ from $g, g^a, g^b$. 
Generic complexity of DLP (Pohlig–Hellman 1978)

- Upshot: If the factorization of $|G|$ is $p_1^{e_1} \cdots p_r^{e_r}$, then one can solve DLP in $O\left(\sum_{i=1}^{r} e_i \cdot (\log |G| + \sqrt{p_i})\right)$ group operations.

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  !! There are many groups where one can solve DLP faster.
Diffie–Hellman’s algebraic properties

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- **exponentiate** exponents: square-and-multiply using \(\otimes\).
- **invert** exponents: \(g^{1/a} = g^{a \cdot \phi(|G|)^{-1}}\) if \(\gcd(a, |G|) = 1\) using \(\otimes\).
Black-box rings

- We interpret $g^a$ as labels for the hidden elements $a$.
- With a CDH oracle we can perform arbitrary ring operations (+,−,⋅,/ ) on these hidden representations.
- Notation: Write $\lceil a \rceil$ for the hidden element $g^a$. 
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We mostly care about **black-box fields**:
For discrete logarithms, it’s sufficient to consider **prime-order** $g$. 
First result: den Boer (1988)

Let $G = \mathbb{F}_p^*$, write $R = \mathbb{Z}/|G| = \mathbb{Z}/(p - 1)$, and suppose $|R^*| = \varphi(p - 1)$ is smooth. Then CDH is polynomial-time equivalent to DLP in $\mathbb{F}_p^*$. 

Proof idea: Solve a DLP in the exponents $R^*$ to find a representation of $\lceil a \rceil$ as a power of some known $\lceil h \rceil$, then recompute $a$ in the clear.

Proof:

- Suppose (for simplicity) that $R^*$ is cyclic with a generator $h$.
- Encode $h$ to a black-box element $\lceil h \rceil$ of $R^*$.
- Solve the DLP $(\lceil h \rceil, \lceil a \rceil)$ in the hidden version of $R^*$ using Pohlig–Hellman. We get $k \in \mathbb{Z}$ such that $g^a = g^{h^k}$.
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Observation:
There is nothing special about using $R^*$ in the exponents; in principle anything expressible as field operations works.
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This is known as an auxiliary group:
A smooth-order algebraic group over the black-box field.
Slightly late ‘about me’ slide

I played too many hacking competitions in $[2013; +\infty)$. 
Example auxiliary group: $\mathbb{F}_{p^2}^*$

A challenge I made for a CTF last year:

Solve a DLP $(g, g^a)$ in a black-box group of order $p = 2^{48} - 5297$ using at most $2^{14}$ queries to $\text{mul}$, $\text{inv}$, and $\text{dhp}$.
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```python
p = 2**48 - 5297  # |G| = p
mul = lambda x,y: (x+y)%p  # g^x * g^y = g^(x+y)
inv = lambda x: (-x)%p  # (g^x)^-1 = g^(-x)
dhp = lambda x,y: x*y%p  # enjoy your oracle!

aes = AES.new(os.urandom(16), AES.MODE_ECB)
enc = lambda x: aes.encrypt(x.to_bytes(16, 'big')).hex()
dec = lambda y: int.from_bytes(aes.decrypt(bytes.fromhex(y)), 'big')

g, a = 1, random.randrange(p)
print(enc(g), enc(y))

for _ in range(2**14):
    q = input().strip().split()
    if q[0] == 'mul': print(enc(mul(dec(q[1])), dec(q[2])))
    if q[0] == 'inv': print(enc(inv(dec(q[1])))
    if q[0] == 'dhp': print(enc(dhp(dec(q[1])), dec(q[2])))

if int(input()) % p == a: print(open('flag.txt').read())
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Intended solution: next slide.
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**Better solution:** [https://sasdf.cf/ctf/writeup/2018/hxp/crypto/blinder_v2/]

- Notice $p + 1 = 2^4 \cdot 3 \cdot 5 \cdot 59 \cdot 281 \cdot 3037 \cdot 23293$ is smooth.
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- Write $\mathbb{F}_{p^2} = \{x + iy : x, y \in \mathbb{F}_p\}$ and fix a generator $h$ of $\mathbb{F}_{p^2}^*$. 
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- Recover $k = \log_{[h]}([a + i]) \mod (p + 1)$ using the oracle.
  
  Thus $h^k = (a + i) \cdot h^{r(p+1)}$ for some $r \in \mathbb{Z}$.
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- Recover $k = \log_{\lfloor h \rfloor}(\lfloor a + i \rfloor) \pmod{(p + 1)}$ using the oracle. Thus $h^k = (a + i) \cdot h^{r(p+1)}$ for some $r \in \mathbb{Z}$.
- Observe that $h^{p+1} \in \mathbb{F}_p^*$, so $h^k = ca + ci$ for some $c \in \mathbb{F}_p^*$.
  $\Rightarrow$ Divide $h^k$ by its $i$-coefficient to obtain $a + i$, hence $a$!
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- Observe that $h^{p+1} \in \mathbb{F}_p^*$, so $h^k = ca + ci$ for some $c \in \mathbb{F}_p^*$. $\implies$ Divide $h^k$ by its $i$-coefficient to obtain $a + i$, hence $a$!

(This uses only $\sim 4500$ queries. Intended solution $\sim 3$ times as many.)
Let $G$ be of prime order $p$, write $R = \mathbb{F}_p$, and suppose $E: y^2 = x^3 + Ax^2 + x \mod \mathbb{F}_p$ has smooth order. Then CDH is polynomial-time equivalent to DLP in $G$. 
Second result: Maurer (1994)

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Proof:

- Find a generator point $G$ on $E$.
- Hope that $a$ is an $x$-coordinate on the curve ($\text{Pr} \approx 1/2$).
  Compute (black-box) the corresponding $y$-coordinate $\lceil y \rceil$, giving a black-box elliptic-curve point $\lceil P \rceil = (\lceil a \rceil, \lceil y \rceil)$.
  (If $\lceil y \rceil^2 \neq \lceil a \rceil^3 + [A] \lceil a \rceil^2 + \lceil a \rceil$, then randomize $\lceil a \rceil$ as $\lceil a' \rceil = \lceil a \rceil + \lceil \delta \rceil$ and retry.)
- Solve the (black-box) DLP ($\lceil G \rceil, \lceil P \rceil$) via Pohlig–Hellman. We get $k \in \mathbb{Z}$ such that $(a, \eta) = \lceil k \rceil G$.
- Simply compute $a$ as the $x$-coordinate of $\lceil k \rceil G$. 

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Are there always such $E$?
Unknown in general, but likely.
People have constructed some for ‘common’ groups $G$.

$\implies$ For all practical purposes, DLP is equivalent to CDH.
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...and they lived happily ever after??
Shor’s algorithm (1994)

Shor’s algorithm **breaks** all group-based DH instantiations.
Shor’s algorithm (1994)

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Let $S$ be some finite set and

$$f : \mathbb{Z}^n \rightarrow S$$

a map with an unknown period lattice $\Lambda \subseteq \mathbb{Z}^n$, such that

$$f(v + \tau) = f(v)$$

if and only if $\tau \in \Lambda$. 
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\[ f(v + \tau) = f(v) \]
if and only if $\tau \in \Lambda$.

Given such $f$ and some size constraints on $\Lambda$, Shor’s algorithm recovers a **basis of $\Lambda$** in polynomial time.
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...is a quantum algorithm for **period finding**.

**Application:**

For $G$ be a cyclic group and $(g, h = g^a)$ a DLP instance in $G$, the (publicly computable) function

$$f : \mathbb{Z}^2 \rightarrow G$$

$$(x, y) \mapsto g^x \cdot h^y$$

has period lattice $\Lambda = \langle (a, -1) \rangle \subseteq \mathbb{Z}^2$, which Shor can recover.
And now...

For something totally different.
Diffie–Hellman from group actions (2006)

Let $G$ be a group, $X$ a set. A group action of $G$ on $X$ is a map

$$*: G \times X \rightarrow X$$

such that $\text{id} * x = x$ and $(g \cdot h) * x = g * (h * x)$.

Example: CSIDH [“si:­saId” (2018)] [joint w / Castryck, Lange, Martindale, Renes]
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This suggests an obvious Diffie–Hellman scheme:
Let $G$ be finite and commutative and fix $x \in X$.

- **Private keys**: group elements $a, b \in G$.
- **Public keys**: the elements $a * x, b * x \in X$.
- **Shared secret**: the element $a * (b * x) = b * (a * x)$. 

This is not in general broken by Shor!

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Example: CSIDH ['siːˌsɔːd] (2018) [joint w/ Castryck, Lange, Martindale, Renes]
The implicit group

Just like before, we get an implicit structure on the public keys.
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Just like before, we get an implicit structure on the public keys. However, crucially, the operation $g^a \cdot g^b = g^{a+b}$ is lost.

$\implies$ We only get a black-box group rather than a ring or field.
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Rather than a ring or field.

Anyone can...

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- **compose**: $\lceil a \rceil \cdot \lceil b \rceil = \text{shared-secret}(x, a \ast x, b \ast x)$. 
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Anyone who can solve CDH can...

- **compose**: \( \lceil a \rceil \cdot \lceil b \rceil = shared\_secret(x, a \cdot x, b \cdot x) \).
- **exponentiate**: square-and-multiply using \( \mathbin{\mathcal{G}} \).
- **invert**: \( \lceil a^{-1} \rceil = \lceil a \rceil |G|^{-1} \) using \( \mathbin{\mathcal{G}} \).
Our result (2018) [joint w/ Galbraith, Smith, Vercauteren]

**Theorem.** There is a polynomial-time quantum equivalence between the CDH and DLP problems for group actions.
Our result (2018) [joint w/ Galbraith, Smith, Vercauteren]

**Theorem.** There is a polynomial-time quantum equivalence between the CDH and DLP problems for group actions.

**Proof:**

- Compute a set of generators $g_1, \ldots, g_r \in G$.
- Apply Shor’s algorithm to the map
  $$f : \mathbb{Z}^r \times \mathbb{Z} \rightarrow X$$
  $$(x_1, \ldots, x_r, y) \mapsto (g_1^{x_1} \cdots g_r^{x_r}) \ast \lfloor a \rfloor^y.$$  

- Any **period vector** of the form $(x_1, \ldots, x_r, 1)$ yields the desired element $a = g_1^{-x_1} \cdots g_r^{-x_r}$.  

Can we get similar results in the group-action setting if the CDH oracle $(x, a \ast x, b \ast x) \mapsto ab \ast x$ is unreliable?

Classical case: Yes, by repeatedly blinding the inputs, unblinding the outputs, and using majority vote.

Here: Exponentially many queries in superposition; do we need all of them to be correct?
Can we get similar results in the group-action setting if the CDH oracle \((x, a \ast x, b \ast x) \mapsto ab \ast x\) is unreliable?

**Likely.** But we haven’t worked this out yet. And it might turn out to be impossible.

So probably it’s best if you forget about it.

At least for the time being.

Until we’ve worked it out.

Hopefully soon.
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Thank you!