Cryptography
&
Quantum Computers

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Technische Universität München

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This talk

Why cryptography?

The quantum threat

Post-quantum everything

Highlight: Isogenies

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The internet

The ARPANET in December 1969

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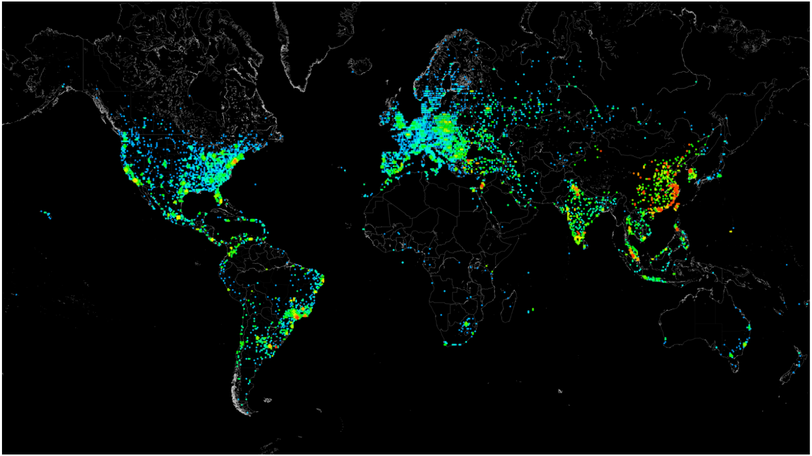

The internet

...is a giant computer network run by not necessarily trustworthy strangers.

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- Existential threat: **CRITICAL INFRASTRUCTURE.**
  - Even airgapped systems are at risk: Firmware updates...

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 This mimics the *intended* properties of a "real" signature.

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 Analogy: An open padlock for which Bob has the key.

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(Notice how this constitutes an important prerequisite for the development of cryptography as a science.)

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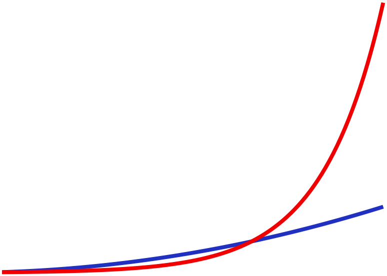

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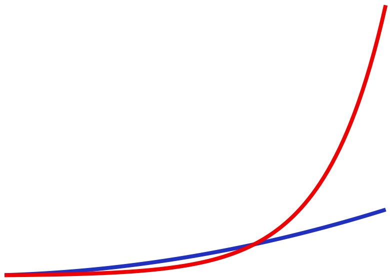

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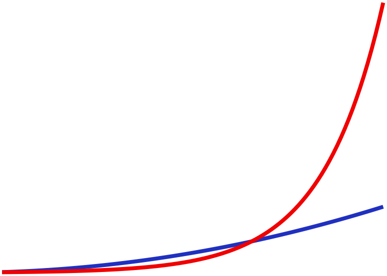

- Great source of hard problems: Algebra!
  Finite fields, elliptic curves, number fields, class groups, ...
  - Key feature: These objects have a lot of useful structure.
  - Sweet spot: just enough to make things *functional* but *secure*.

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- Theory: If nontrivial cryptography is secure, then $\mathbf{P} \neq \mathbf{NP}$.
  Reality: Does it matter? Is an $O(n^{666})$ algorithm really "tractable"?

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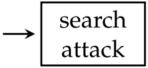

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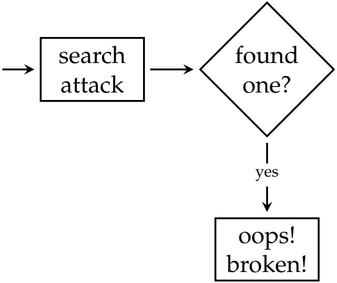

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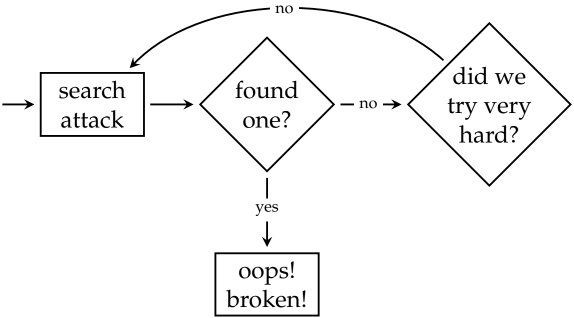

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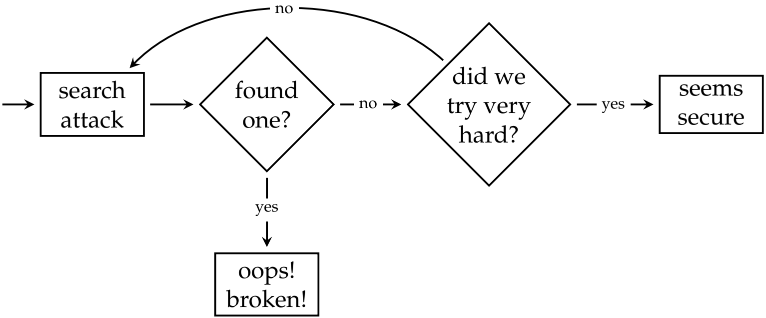

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  $\rightsquigarrow$ Theory of **average-case hardness**.

- The problems mainly used in contemporary public-key cryptography are in fact unlikely to be **NP**-hard!

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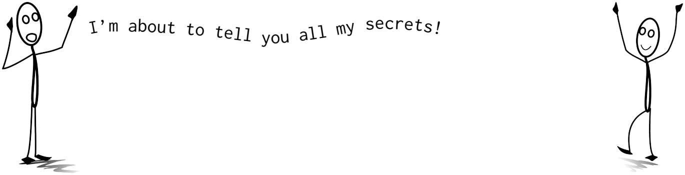

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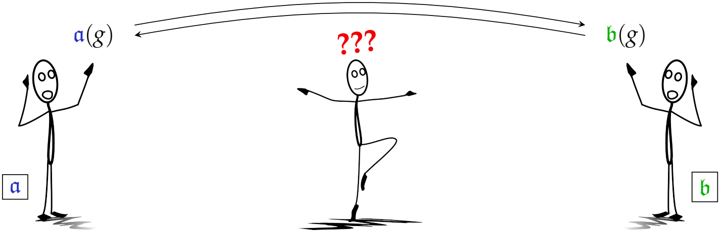

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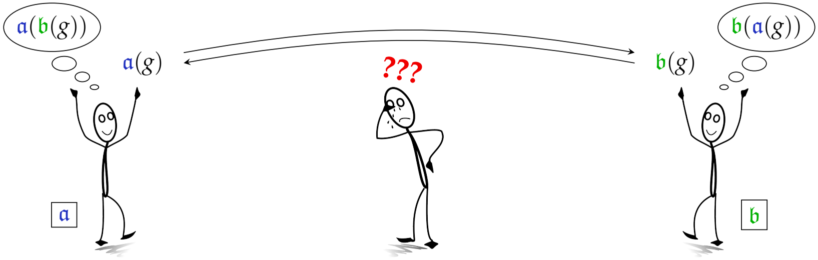

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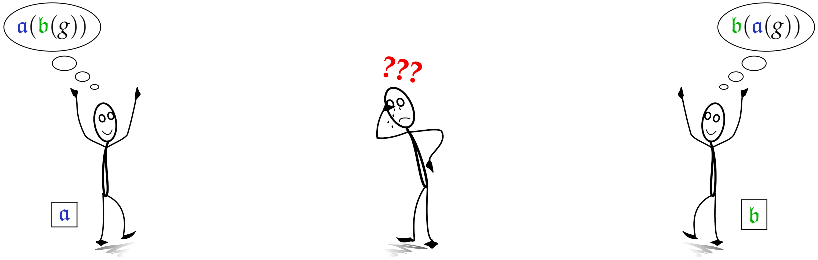

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  This "works" ($(g^a)^b = (g^b)^a$), but with real numbers it's either clearly impossible to do this efficiently or insecure.
- Excellent idea: Do it in finite algebraic structures.
  This still "works", and can be secure and efficient.

Diffie–Hellman key agreement (1976)

Forever fixed, public system parameters:

- A large prime p of the form $p = 2\ell + 1$ with ℓ prime.
- An element $g \in \mathbb{F}_p^\times$ of order ℓ.

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Alice	public	Bob
$a \leftarrow \{0, \ldots, \ell-1\}$.		$b \leftarrow \{0, \ldots, \ell-1\}$.
g^a	$\longrightarrow$ / $\longleftarrow$	g^b
Compute $(g^b)^a$.		Compute $(g^a)^b$.

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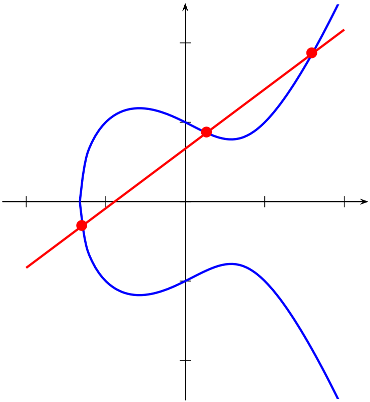

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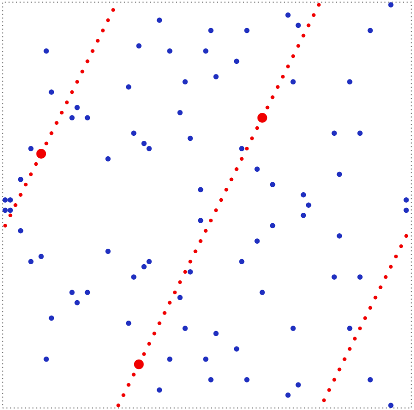

The elliptic curve $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

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Highlight: Isogenies

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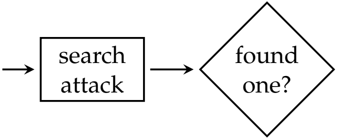

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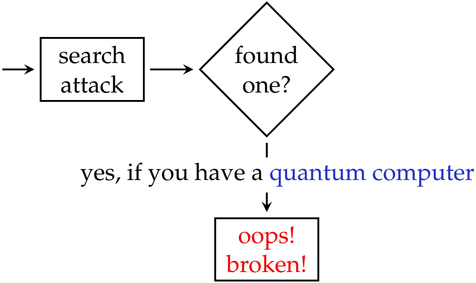

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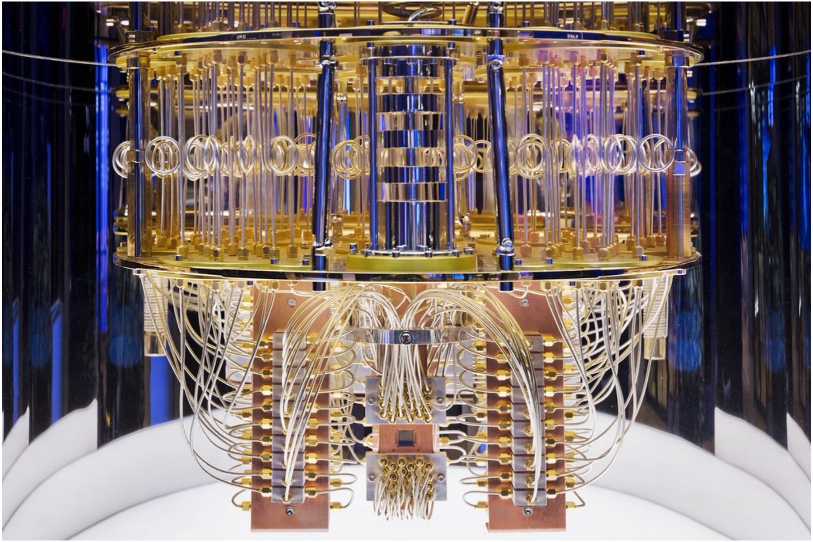
⇝ Quantum computers are just "the next evolution" of using an increasingly bigger share of physics to compute things.



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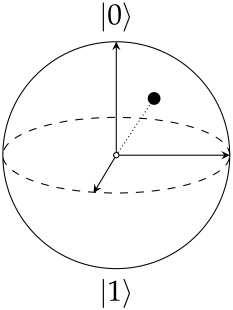

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"Quantum states are like a box of chocolates.
You never know what you're gonna get."

— F. Gump, probably

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This does **not** mean that "quantum computers can simply search through all secret keys simultaneously"!

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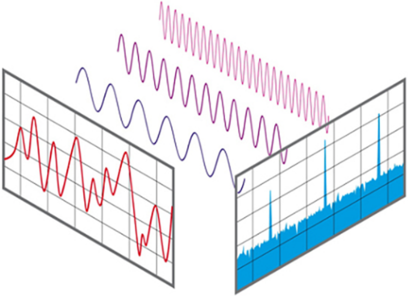
  - !! Polynomial-time complexity. (More on the next slide.)
- Kuperberg's algorithm: Given two functions $f_1, f_2 : G \to S$ such that $\exists!\, s \in G$ with $f_2(x) = f_1(x+s)$ for all x, find that s.
  - !! Subexponential complexity: from $|G|^{O(1)}$ to $2^{O(\sqrt{\log|G|})}$.

Shor's algorithm

Key idea: **Q**uantum **F**ourier **T**ransform.



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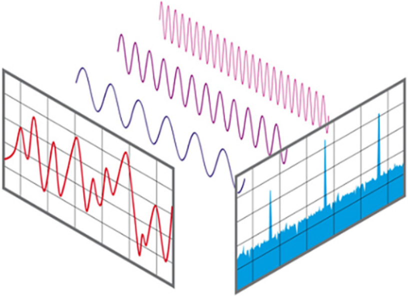

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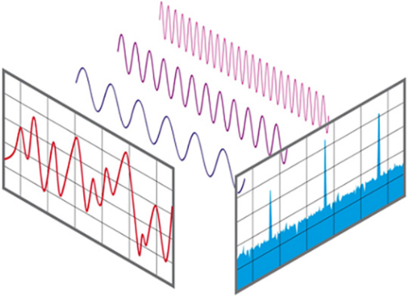

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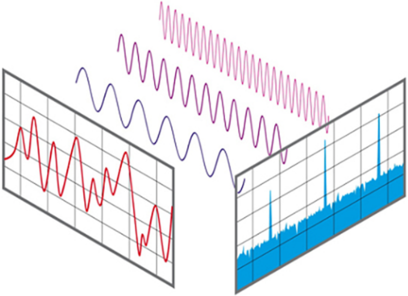

- Prepare a quantum state encoding a preimage set $f^{-1}(y)$.
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- Then measure to get a random period vector. Repeat.

Shor vs. DLP

The **d**iscrete **l**ogarithm **p**roblem: Given g and $h = g^a$, find a.

(Here g, h are elements of a finite group G and a is an integer.)

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Exponential speedup.

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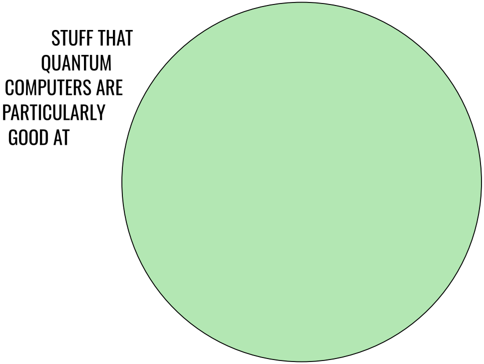

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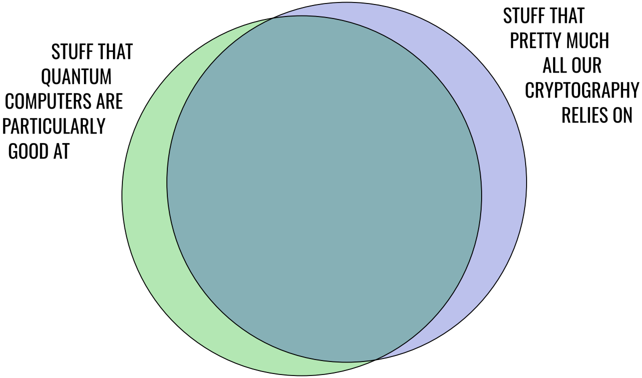

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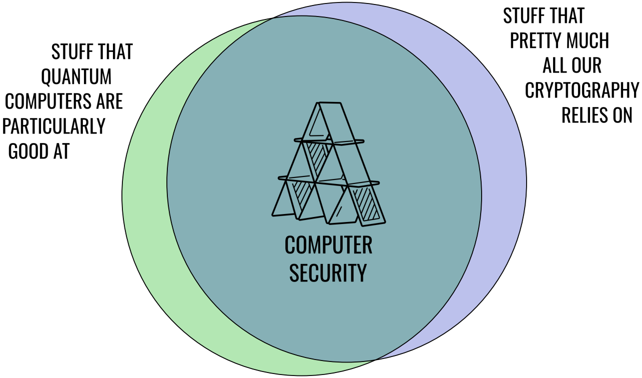

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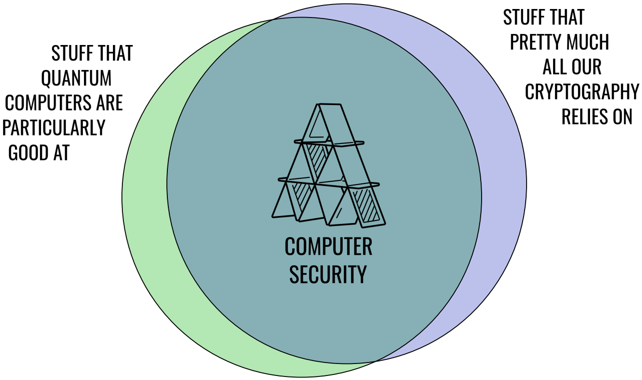

- Note: Public-key cryptography sustains much more damage from quantum attacks than symmetric cryptography.

This talk

Why cryptography?

The quantum threat

Post-quantum everything

Highlight: Isogenies

Post-quantum cryptography (PQC)

...substitutes quantum-weak building blocks by
quantum-resistant alternatives.

Note on "quantum cryptography"

- Post-quantum cryptography is not to be confused with "quantum cryptography".

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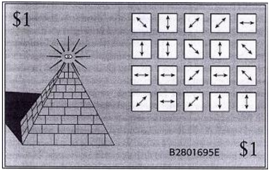

Note on “quantum cryptography”

RÉPUBLIQUE FRANÇAISE
Liberté
Égalité
Fraternité

Federal Office for Information Security

General Intelligence and Security Service
Ministry of the Interior and Kingdom Relations

SWEDISH ARMED FORCES

Position Paper on Quantum Key Distribution

French Cybersecurity Agency (ANSSI)

Federal Office for Information Security (BSI)

Netherlands National Communications Security Agency (NLNCSA)

Swedish National Communications Security Authority, Swedish Armed Forces

Note on “quantum cryptography”

Executive summary

Quantum Key Distribution (QKD) seeks to leverage quantum effects in order for two remote parties to agree on a secret key via an insecure quantum channel. This technology has received significant attention, sometimes claiming unprecedented levels of security against attacks by both classical and quantum computers.

Due to current and inherent limitations, QKD can however currently only be used in practice in some niche use cases. For the vast majority of use cases where classical key agreement schemes are currently used it is not possible to use QKD in practice. Furthermore, QKD is not yet sufficiently mature from a security perspective. In light of the urgent need to stop relying only on quantum-vulnerable public-key cryptography for key establishment, the clear priorities should therefore be the migration to post-quantum cryptography and/or the adoption of symmetric keying.

This paper is aimed at a general audience. Technical details have therefore been left out to the extent possible. Technical terms that require a definition are printed in italics and are explained in a glossary at the end of the document.

The post-quantum zoo

- PQC uses alternative hardness assumptions
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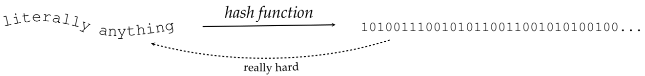
Hash-based signatures

Hash functions are random-looking functions that compress arbitrary data to short bitstrings. They should be hard to invert.



An individual can tie a hash value to their identity and later identify themself by revealing the corresponding input.
Selectively revealing inputs depending on a message leads to a signature scheme.

The post-quantum zoo

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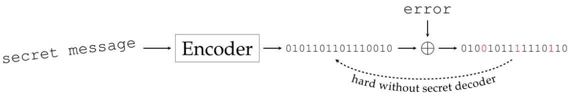
Code-based crypto

Main application: **Encryption**.

Underlying problem: Correct errors in a codeword of a random-looking code.



Oldest proposal: McEliece 1978. Still *essentially unbroken* [2].

The post-quantum zoo

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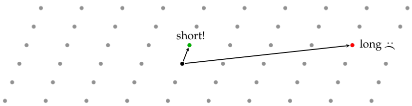
Lattice-based crypto

Main applications: **Encryption**, **signatures**, and beyond.

Underlying problem: Find short vectors in a discrete additive subgroup of $\mathbb{R}^n$.



The post-quantum zoo

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Multivariate crypto

Main application: **Signatures**.

Underlying problem: Solve systems of quadratic equations over a finite field.

$$
\begin{aligned}
10x^2 + 15z^2 + 19xy + 7xz + 27yz + 20x + y &\equiv 14 \pmod{31}\\
25x^2 + 30y^2 + 17z^2 + 30xy + 23xz + 27yz + 15x + 4y + 16z &\equiv 5 \pmod{31}\\
15x^2 + 9y^2 + 11z^2 + 18xy + 24xz + 16yz + 28x + 9y + 3z &\equiv 6 \pmod{31}\\
27x^2 + 10y^2 + 17z^2 + 7xz + 28yz + 4x + 13y + 27z &\equiv 12 \pmod{31}
\end{aligned}
$$

The post-quantum zoo

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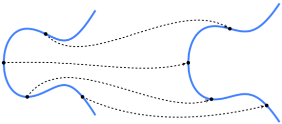
Isogeny-based crypto

Main application: **Key exchange**, **signatures**.

Underlying problem: Find an isogeny between two elliptic curves.
An *isogeny* is a surjective group homomorphism given by rational functions.



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pre-quantum	post-quantum
6ee2da7b68b7a997e062d09d94c1c76de61b5c260a35273713ddcc29e09ac840	45c83435071624067d69587335b97bf564929709c8825a004b028ae09c40980a 07e8d4bd604527ee221e8bac67d34cbe762c26df8453aae8b8c82b59c51a8552 6aba8ddc4b5f63cf69a5b367d3153e460f497a209c495fca318862d6a5780086 5479a006012d82f7212b40284d310e01bcb11e122c1fd303e441807849a7ea47 976a99abb7ccc4b674ad66f68eca195789b277d23c3d67bc418ca7c908b21e53 984983ba0205e4689000ace97238b3699016fa95e7a3a59cec0be81363852756 2fa9bf10d715e7505f6e1c1433521a918a7df52760a0d8a9549569f10827c423 cddff82aae01a90111395487b9c82b7b5a7978d789679e66b75087bfbff0569f c94e94f93531b721315926388431f2a36ae0f701bac254befb437c58641d4560 c8738a98f30918945db0a6900ad2c2abfc3e0f4786a4555639d84dcdd031d8f0 508d8c774d68298bcac4f42c6a7ff585af491fa7d7c3bbb41727699ebb315c43 7b210d42626ebc66c916af1f3515374314e4f40309ca7289c7bc51c301d8180e dc792d4dd44c41b77bd47a972d8434a9f03bb3954236ec422be0c8e991a79af2 86b6a7c459a95ed44868ed8052f2db0f3741710228979507cff961564882b5ea 19515ee00d657c7141e9b05f9a24136a2f915620b664404b5397cc7842748973 d0716cc273b528d51383a63fc8a3c4a3b1a8bc965775d750add6996c929e29f4 1e42362a759baa76f5a3dc0552f1d83195960e45837901494a87f2a6dc3b5d8b 73a9695c1229a0c9bddb0b2d99aa350c6cac657745c1308af354e10595f3682a 34dc26d9d28e2e2c4634aca75e94384700c9c06b1bca348330ac1791fab14190 99cf1288283bab03dca09ab3593cf3b12739cb44c0c04c6b93d1ea831df6bcb8 807aa6aa8cbec64d749a9e47f851c47c6537e196f1fcc4d63b67d29a58e86b9a 72a199cbb793c5084e5bab20bd02289b4aaa64e4c119488531e8a651a3175014 8e1742c5390bb9995c123f3056ad44c476468ded4b88a49130e35b4b00803dd2 4718674ca708e436d5c15ee1d95367c623512653c83b27b41cb308f8c2929b19 3b5487a4ce6401ec27a1605f879e2d9c53bf27e165246401cad7840a077934b8

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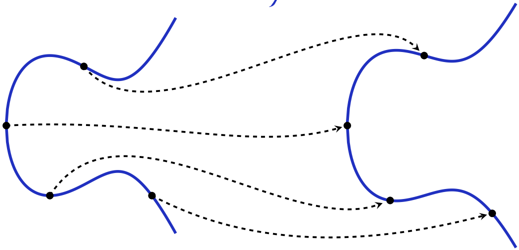
Highlight: Isogenies

Isogenies of elliptic curves

- ...are essentially just *nice maps* between elliptic curves.



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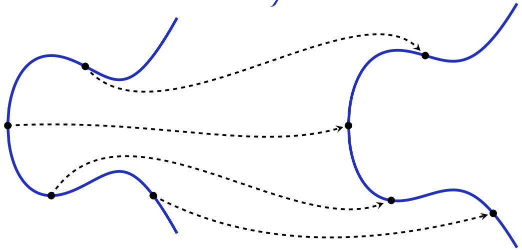
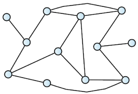


- They are a source of exponentially large graphs.



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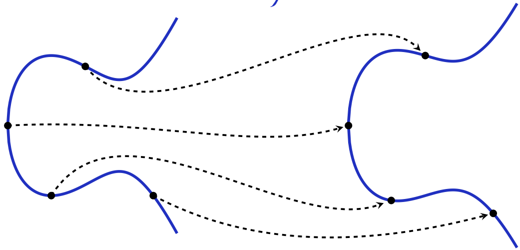

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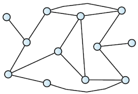
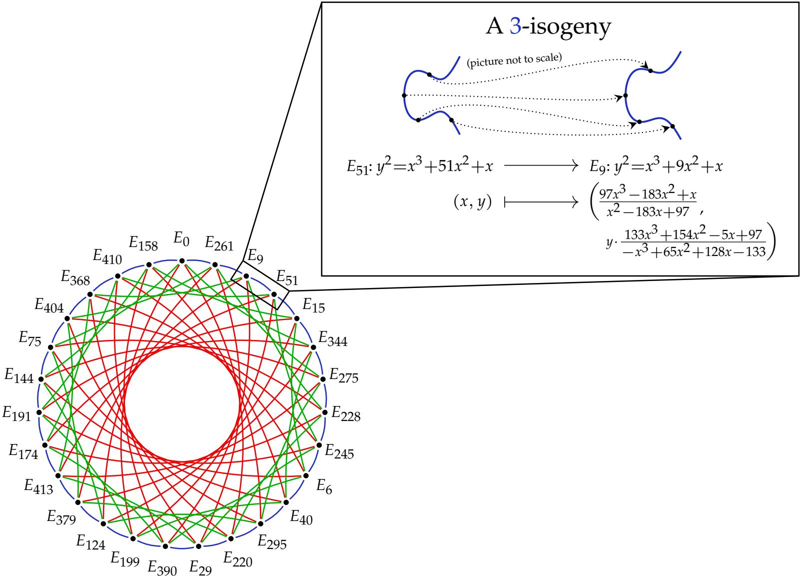


- ...with enough structure to navigate meaningfully!

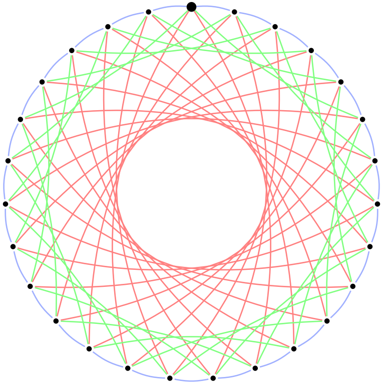
Graphs of elliptic curves



CSIDH [ˈsiːˌsaɪd] key exchange

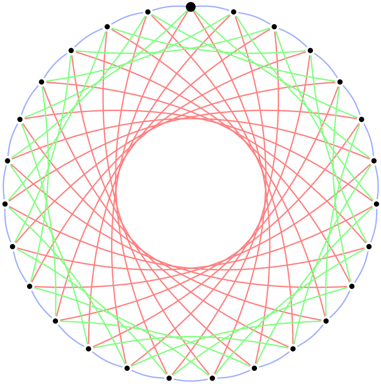
Alice

[+, +, −, −]



Bob

[−, +, −, −]



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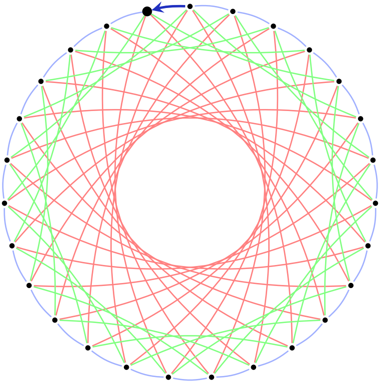

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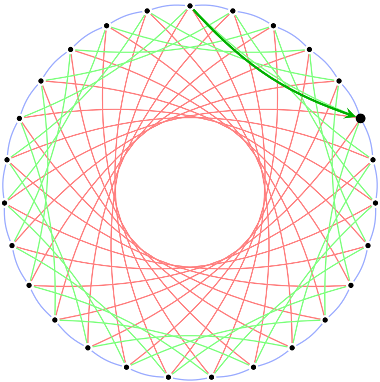

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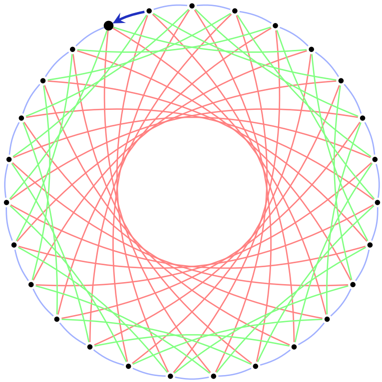

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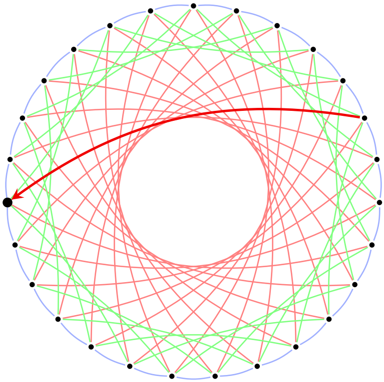

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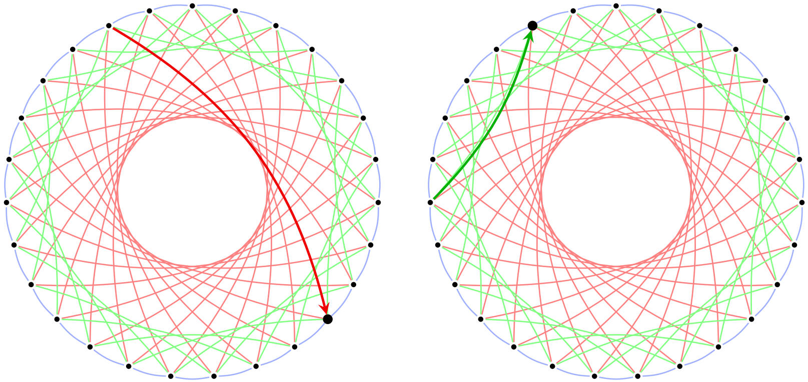

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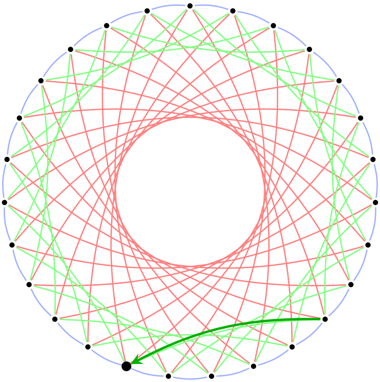

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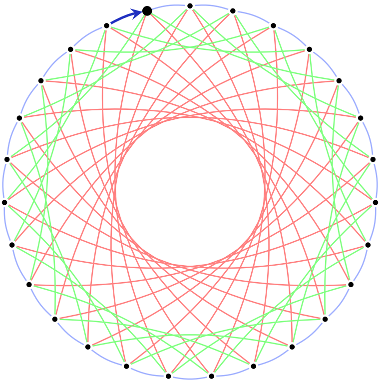

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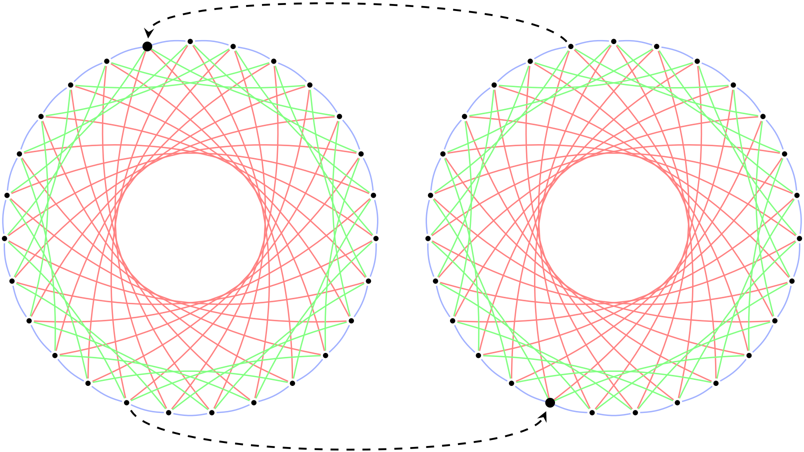

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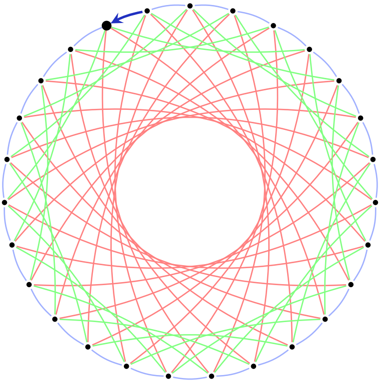

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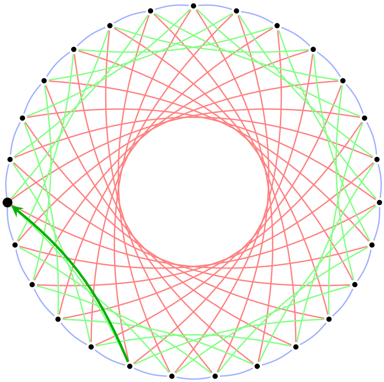

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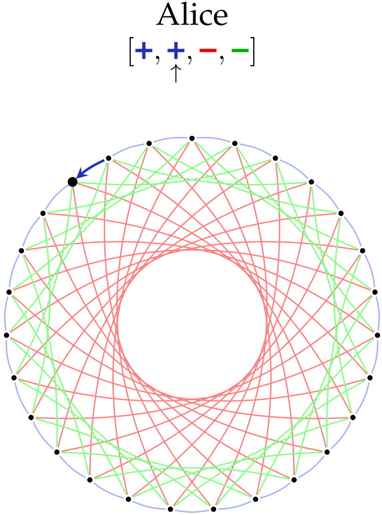

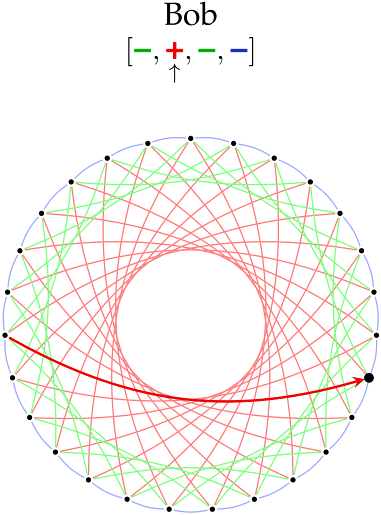

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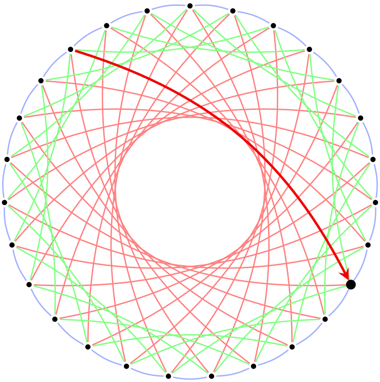
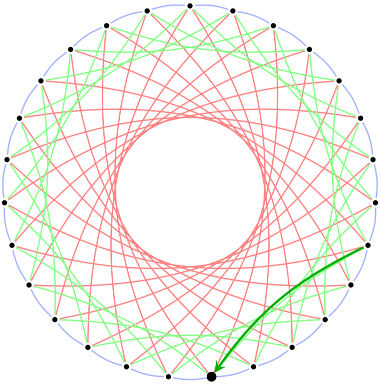


Bob

[−, +, −, −]

↑



CSIDH [ˈsiːˌsaɪd] key exchange

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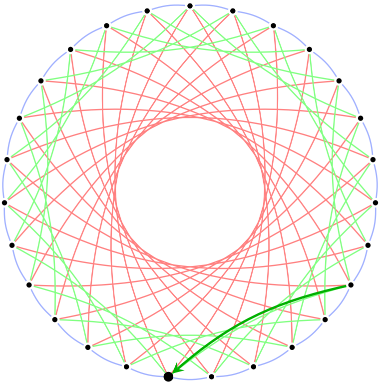
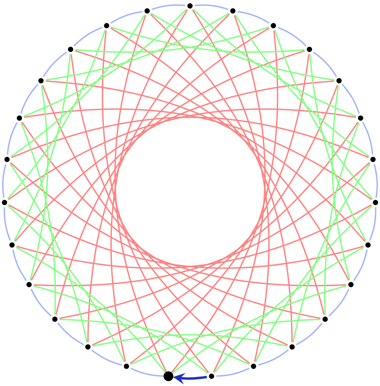
[**+**, **+**, **−**, **−**]
↑

Bob

[**−**, **+**, **−**, **−**]
↑





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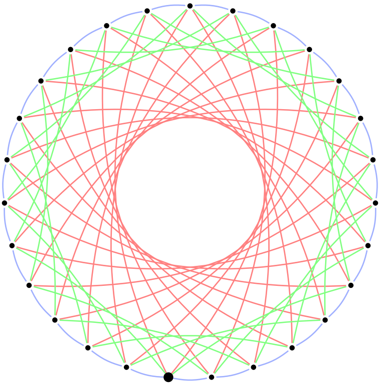

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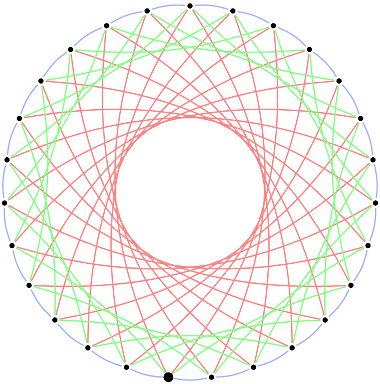
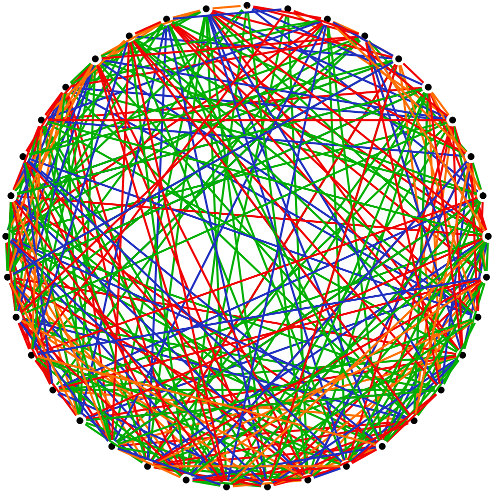


A much more random-looking isogeny graph



SQIsign

Isogeny graphs are not random graphs.

Lots of useful structure looming in the background.

SQIsign

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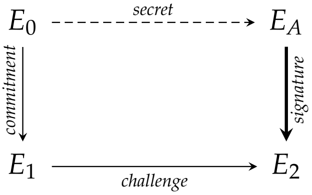
SQIsign is a signature scheme based on this one-wayness.



This talk

Why cryptography?

The quantum threat

Post-quantum everything

Highlight: Isogenies

Main goal: Solid foundations for computer security

- **Cryptanalysis!**
  - Algorithmic advances
  - Low-level programming (CPUs, GPUs)
  - Quantum algorithms

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  $\longrightarrow$ Open-source software

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  - Low-level programming (again)
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- Information security in general
  - Memory corruptions, reverse engineering, web hacking, ...