How to not break SIDH $\frown$

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What is this all about?

Diffie–Hellman key exchange '76

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today also elliptic curves)
- an element $g \in G$ of prime order p

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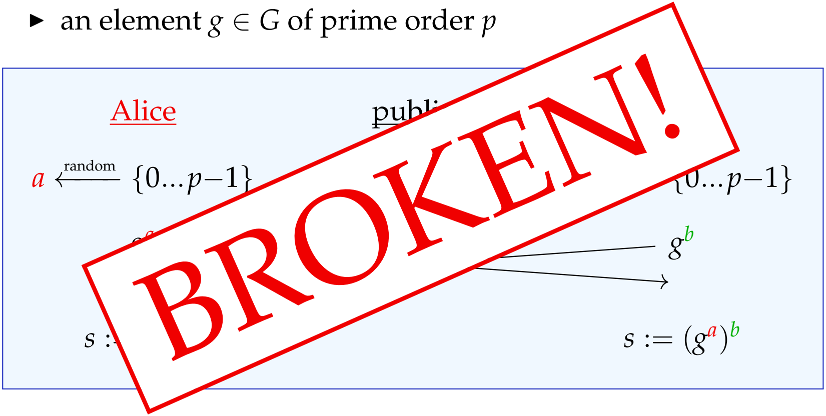

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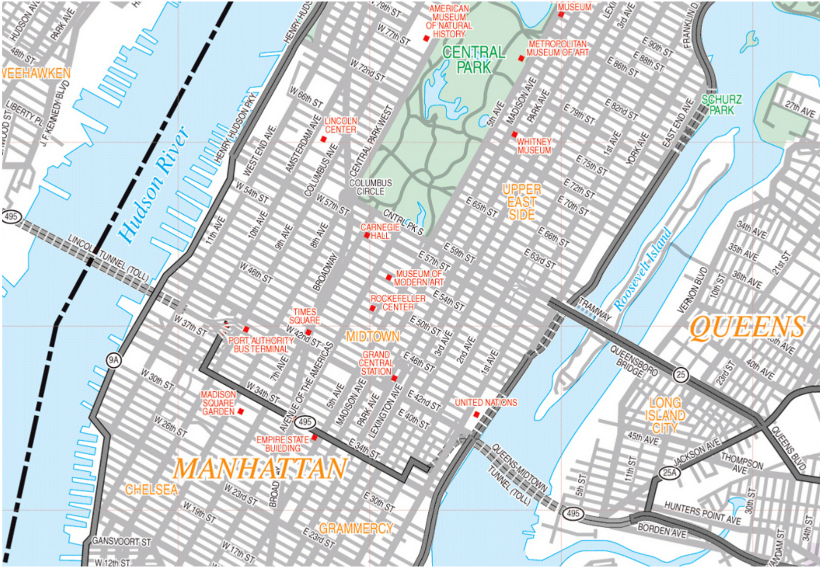

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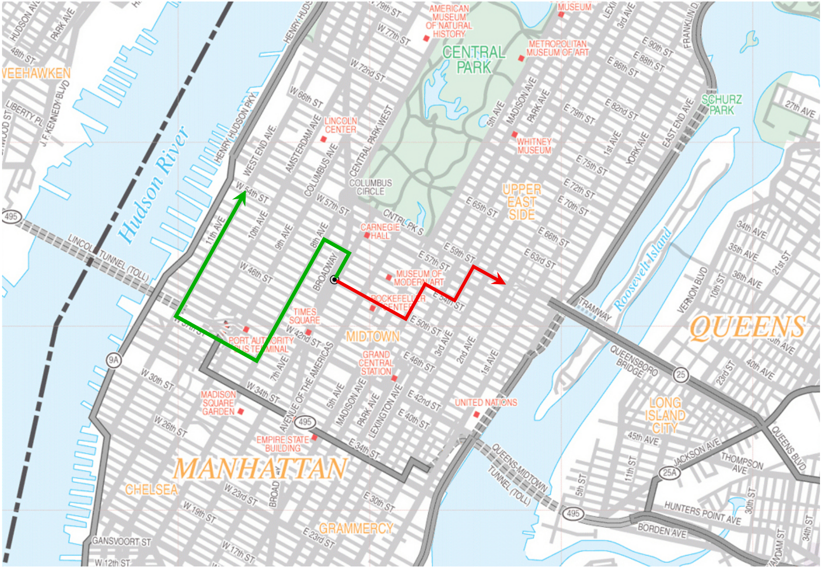

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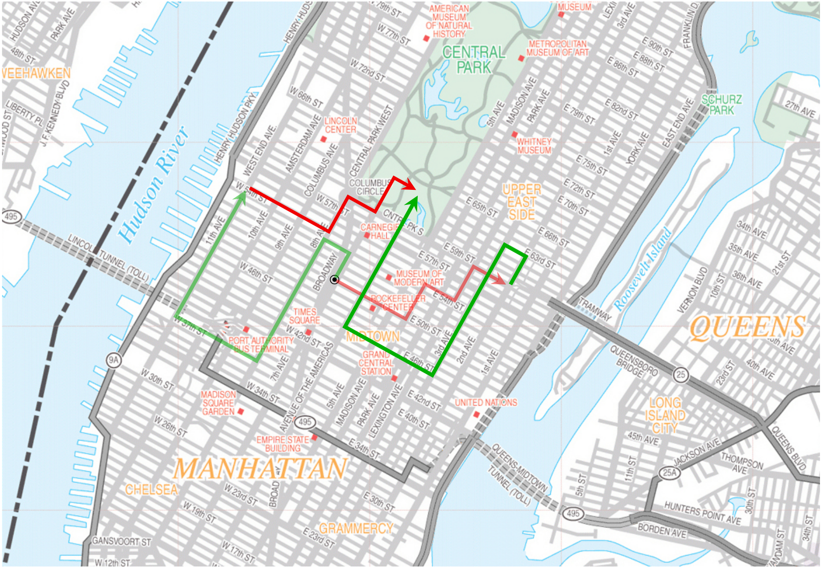

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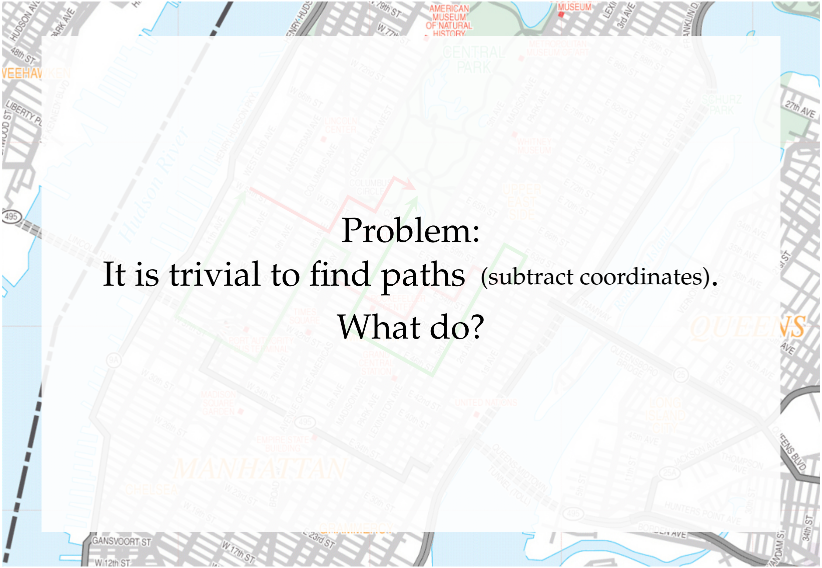

Problem:

It is trivial to find paths (subtract coordinates).

What do?

Big picture

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  That is: some *well-behaved* 'directions' to describe paths. More later.

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  That is: some *well-behaved* 'directions' to describe paths. More later.

It is easy to construct graphs that satisfy *almost* all of these — not enough for crypto!

Stand back!

We're going to do math.

Math slide #1: Elliptic curves *(nodes)*

An elliptic curve (modulo details) is given by an equation

$$E\colon\ y^2 = x^3 + ax + b.$$

A point on E is a solution to this equation *or* the 'fake' point ∞.

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E is an abelian group: we can ‘add’ points.

- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

$$\left(\lambda^2 - x_1 - x_2,\ \lambda(2x_1 + x_2 - \lambda^2) - y_1\right)$$

where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + a}{2y_1}$ otherwise.

do **not** remember these formulas!

Math slide #2: Isogenies *(edges)*

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- given by rational functions.
- a group homomorphism.

The degree of a separable* isogeny is the size of its kernel.

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Example #1: For each $m \neq 0$, the multiplication-by-m map

$$[m] \colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #2: For any a and b, the map $\iota\colon (x,y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}\,.$$

It is an isomorphism; its kernel is $\{\infty\}$.

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Example #3: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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An endomorphism of E is an isogeny $E \to E$, or the zero map.
The ring of endomorphisms of E is denoted by $\mathrm{End}(E)$.

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Each isogeny $\varphi\colon E \to E'$ has a unique dual isogeny $\widehat{\varphi}\colon E' \to E$ characterized by $\widehat{\varphi} \circ \varphi = \varphi \circ \widehat{\varphi} = [\deg \varphi]$.

Math slide #3: Fields of definition

Until now: Everything over the algebraic closure.

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For E defined over k, let $E(k)$ be the points of E defined over k.

Math slide #4: Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable isogeny $\varphi_G \colon E \to E'$ with kernel G.

The curve E' is denoted by E/G. (cf. quotient groups)

If G is defined over k, then φ_G and E/G are also defined over k.

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Vélu operates in the field where the points in G live.
$\rightsquigarrow$ need to make sure extensions stay small for desired $\#G$
$\rightsquigarrow$ this is why we use supersingular curves!

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Math slide #5: Supersingular isogeny graphs

Let p be a prime, q a power of p, and ℓ a positive integer $\notin p\mathbb{Z}$.

An elliptic curve $E/\mathbb{F}_q$ is *supersingular* if $p \mid (q + 1 - \#E(\mathbb{F}_q))$.

We care about the cases $\#E(\mathbb{F}_p) = p + 1$ and $\#E(\mathbb{F}_{p^2}) = (p + 1)^2$.

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Let $S \not\ni p$ denote a set of prime numbers.

The supersingular S-isogeny graph over $\mathbb{F}_q$ consists of:

- vertices given by isomorphism classes of supersingular elliptic curves,
- edges given by equivalence classes[1] of ℓ-isogenies ($\ell \in S$),

both defined over $\mathbb{F}_q$.

[1] Two isogenies $\varphi\colon E \to E'$ and $\psi\colon E \to E''$ are identified if $\psi = \iota \circ \varphi$ for some isomorphism $\iota\colon E' \to E''$.

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Components of the isogeny graphs look like this:

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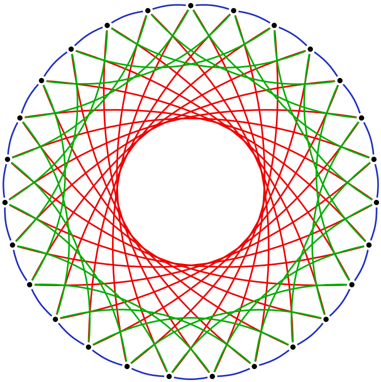

$S = \{3, 5, 7\}$, $q = 419$

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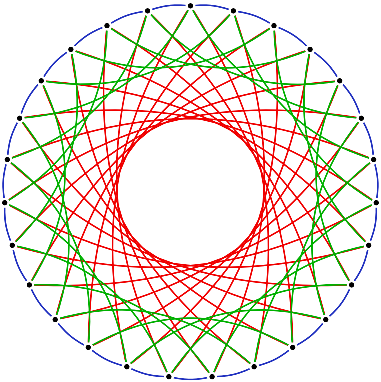

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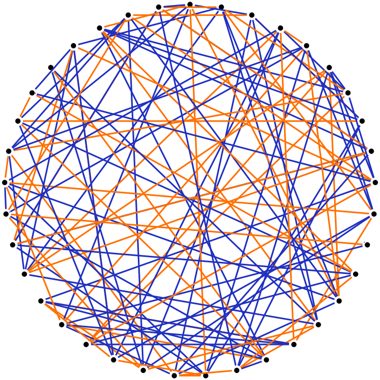
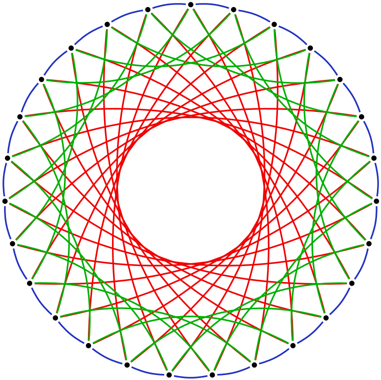


$S = \{2, 3\}$, $q = 431^2$

The beauty and the beast

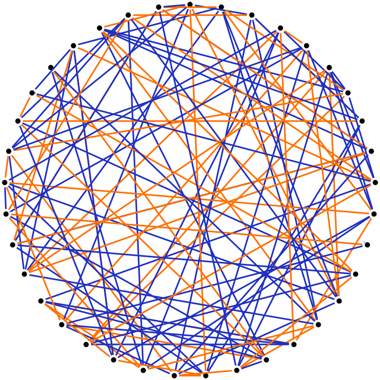
At this time, there are <u>two distinct families</u> of systems:



$q = p$

CSIDH [ˈsiːˌsaɪd]

`https://csidh.isogeny.org`



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SIDH

`https://sike.org`

...we’ll be right back after a short commercial break...

[ˈsiːˌsaɪd]

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Life’s good at the CSIDH!

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Now:
SIDH

(...whose name doesn't allow for nice pictures of beaches...)

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- **SIDH** uses the full $\mathbb{F}_{p^2}$-isogeny graph. No group action!
- Problem: also no intrinsic sense of direction.

"It all bloody looks the same!" — a famous isogeny cryptographer

$\rightsquigarrow$ need extra information to let Alice & Bob's walks commute.

SIDH: High-level view

$$\begin{tikzcd} E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E/A \arrow[d, "\varphi_{B'}"] \\ E/B \arrow[r, "\varphi_{A'}"'] & E/\langle A, B \rangle \end{tikzcd}$$

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- Alice and Bob transmit the values E/A and E/B.
- Alice somehow obtains $A' := \varphi_B(A)$. (Similar for Bob.)
- They both compute the shared secret
  $$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

SIDH's auxiliary points

Previous slide: "Alice <u>somehow</u> obtains $A' := \varphi_B(A)$."

Alice knows only A, Bob knows only φ_B. Hm.

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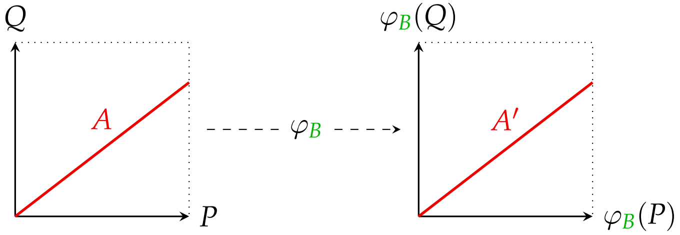

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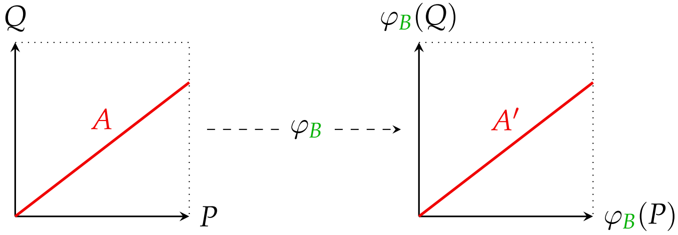

- Alice picks A as $\langle P + [a]Q \rangle$ for fixed public $P, Q \in E$.
- Bob includes $\varphi_B(P)$ and $\varphi_B(Q)$ in his public key.

$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$!

SIDH in one slide

Public parameters:

- a large prime $p = 2^n 3^m - 1$ and a supersingular $E/\mathbb{F}_p$
- bases (P_A, Q_A) and (P_B, Q_B) of $E[2^n]$ and $E[3^m]$

Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \dots 2^n - 1\}$		$b \xleftarrow{\text{random}} \{0 \dots 3^m - 1\}$
$A := \langle P_A + [a]Q_A \rangle$ compute $\varphi_A : E \to E/A$		$B := \langle P_B + [b]Q_B \rangle$ compute $\varphi_B : E \to E/B$
$E/A,\ \varphi_A(P_B),\ \varphi_A(Q_B)$		$E/B,\ \varphi_B(P_A),\ \varphi_B(Q_A)$
$A' := \langle \varphi_B(P_A) + [a]\varphi_B(Q_A) \rangle$ $s := j\big((E/B)/A'\big)$		$B' := \langle \varphi_A(P_B) + [b]\varphi_A(Q_B) \rangle$ $s := j\big((E/A)/B'\big)$

All of the following is 'obvious' to the experts.

We often observe smart people rediscovering and wasting time on these ideas.

Extra points: Information theory

- By linearity, the two points $\varphi_A(P_B), \varphi_A(Q_B)$ encode how φ_A acts on the whole 3^m-torsion.
- Note 3^m is smooth $\rightsquigarrow$ can evaluate φ_A on any $R \in E_0[3^m]$.

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Lemma. If two d-isogenies ϕ, ψ act the same on the m-torsion and $m^2 > 4d$, then $\phi = \psi$.

$\Longrightarrow$ Except for very imbalanced parameters, the public points uniquely determine the secret isogenies.

Extra points: Interpolation?

- Recall: Isogenies are rational maps.
  We know enough input-output pairs to determine the map.

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$\rightsquigarrow$ can't even write down the result without decomposing into a sequence of smaller-degree maps.

- No known algorithms for interpolating and decomposing at the same time.

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"[...] elliptic curves are as close to generic groups as it gets."
— me, 2018

(Exception: pairings, but those are also just bilinear maps.)

Extra points: Effective Tate?

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Theorem. For ell. curves $E, E'/\mathbb{F}_q$ and a prime $\ell \neq p$, the map $\mathrm{Hom}_{\mathbb{F}_q}(E, E') \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{\mathbb{F}_q}(E[\ell^\infty], E'[\ell^\infty])$ is bijective.

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!! We know more: The degree!

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> **Theorem.** For ell. curves $E, E'/\mathbb{F}_q$ and a prime $\ell \neq p$, the map $\mathrm{Hom}_{\mathbb{F}_q}(E, E') \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{\mathbb{F}_q}(E[\ell^\infty], E'[\ell^\infty])$ is bijective.

Read: An isogeny is uniquely defined by how it acts on sufficiently high ℓ^k-torsion.

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- This idea works slightly better for *endo*morphisms (characteristic polynomial constrains to ℓ^2 choices).

Extra points: Petit's endomorphisms (1)

- For typical SIDH parameters, we know endomorphisms ι, π of E_0 such that $\mathrm{End}(E_0) = \left\langle 1, \iota, \frac{\iota+\pi}{2}, \frac{1+\iota\pi}{2} \right\rangle$.

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- Idea: Find $\tau \in R$ of degree $3^m r$; recover 3^m-part from known action; brute-force the $r$emaining part.
$\Longrightarrow$ (details) $\Longrightarrow$ Recover φ_A.

Extra points: Petit's endomorphisms (2)

- Petit uses endomorphisms $\tau \in R$ of the form
$$\tau = a + \varphi_A(b\iota + c\pi + d\iota\pi)\widehat{\varphi_A}\,,$$
where $\deg \iota = 1$ and $\deg \pi = \deg \iota\pi = p$. Hence
$$\deg \tau = a^2 + 2^{2n}b^2 + 2^{2n}pc^2 + 2^{2n}pd^2\,.$$
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$\Longrightarrow$ Unless $3^m \gg 2^n$, there is no hope to find τ with $3^m \mid \deg\tau$ and $\deg\tau/3^m < 2^n$.

Extra points: Summary

- Same problem all over the place:
  There seems to be no way to obtain *anything* from the given action-on-3^m-torsion except what's given.

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- Life sucks.

  ¯_(ツ)_/¯

The pure isogeny problem

Fundamental problem: given supersingular E and $E'/\mathbb{F}_{p^2}$ that are ℓ^n-isogeneous, compute an isogeny $\phi : E \to E'$.

The pure isogeny problem

Example

Choose

$$E/\mathbb{F}_{431} : y^2 = x^3 + 1 \quad \text{and} \quad E'/\mathbb{F}_{431} : y^2 = x^3 + 291x + 298.$$

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These elliptic curves are $2^2 = 4$-isogenous. Problem: compute an isogeny $f : E \to E'$.

The kernel of $f : E \to E'$ is generated by a point $P \in E(\overline{\mathbb{F}_p})$ of order 4.

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- Solution (b): try all three possible order 2 kernels from both E and E' and check when the codomain is the same.

Solution (b) is meet-in-the-middle: complexity $\tilde{O}(p^{1/4})$.

Exploiting subgraphs

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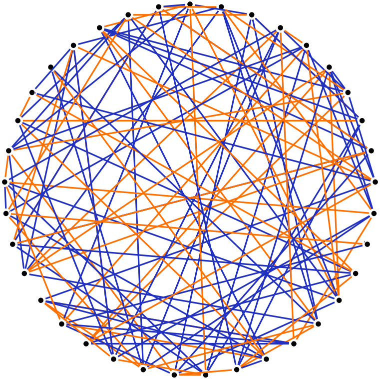

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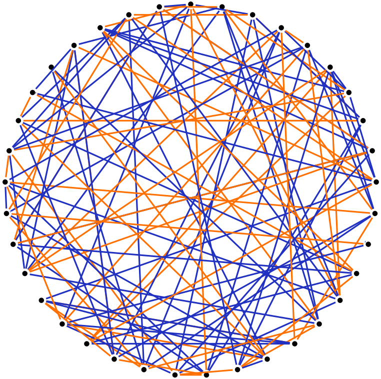

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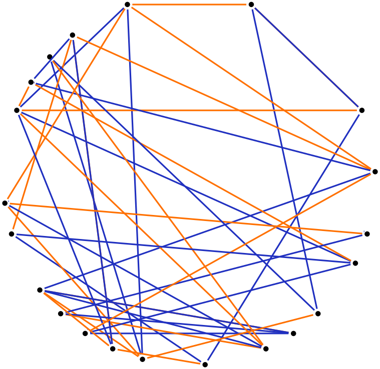

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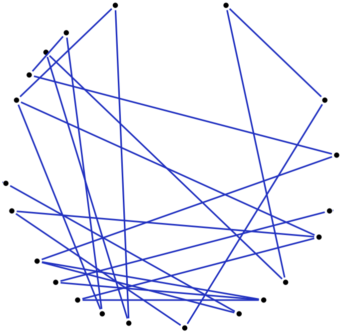

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nodes up to $\overline{\mathbb{F}_p}$-isomorphism

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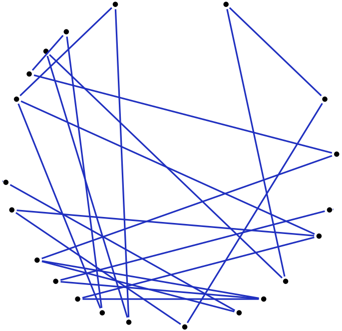

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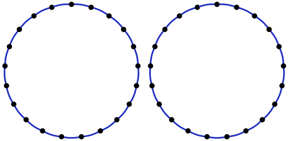

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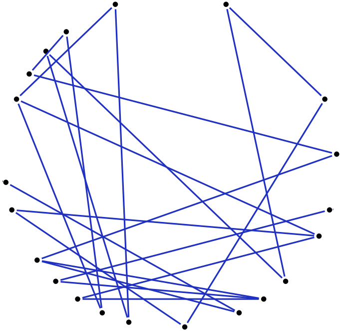

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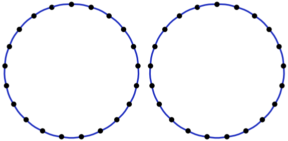

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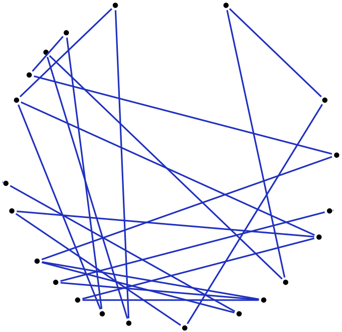

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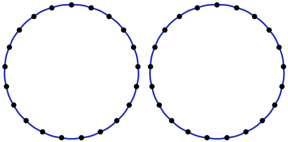

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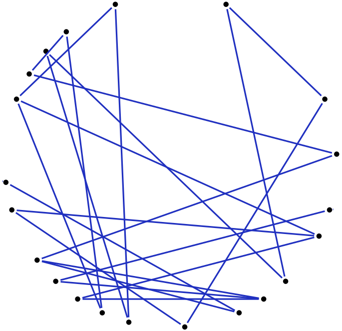

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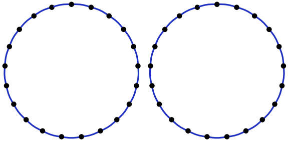

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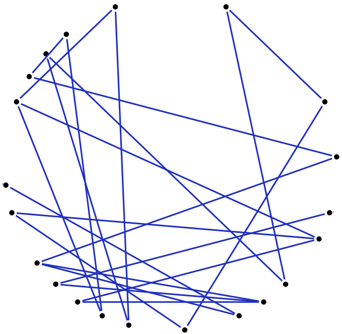

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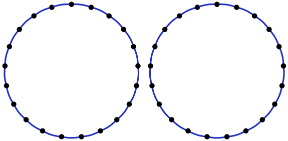

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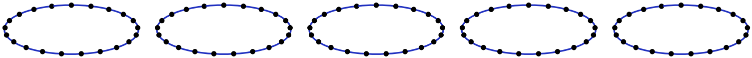
(Delfs-Galbraith, Biasse-Jao-Sankar)

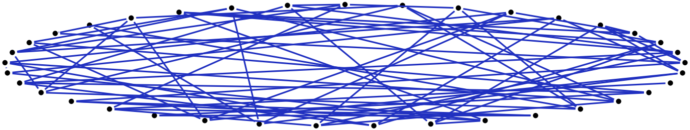
More graphs defined over $\mathbb{F}_p$



From 1-dimensional $E/\mathbb{F}_{p^2}$,

construct 2-dimensional $W(E)/\mathbb{F}_p$

'Weil restriction'



This picture is very unlikely to be accurate.

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- The associated graph of 2-dimensional objects is (heuristically) $O(\sqrt{p})$ cycles of length $O(\sqrt{p})$.
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More equivalent categories: lifting to $\mathbb{C}$

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Here computing isogenies is easy!

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{Non-supersingular elliptic curves defined over } \mathbb{F}_q \\ \text{with } \operatorname{End}(E) = R \end{array} \right\}$$

Here computing isogenies is harder.

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- Computing the equivalence is slow.
- Finding a non-scalar endomorphism is hard.
- If you can find non-scalar endomorphisms, SIDH is probably already broken by earlier work (Kohel-Lauter-Petit-Tignol and Galbraith-Petit-Shani-Ti).

¯_(ツ)_/¯

Thank you!