Isogeny Group Actions

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ECC 2024 Autumn School, Taipei, 29 October 2024

Crypto(graphy) on graphs

Diffie–Hellman key exchange 1976

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today elliptic curves)
- an element $g \in G$ of prime order q

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Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \ldots q-1\}$		$b \xleftarrow{\text{random}} \{0 \ldots q-1\}$
g^a	$g^a \rightarrow$ Bob, $g^b \rightarrow$ Alice	g^b
$s := (g^b)^a$		$s := (g^a)^b$

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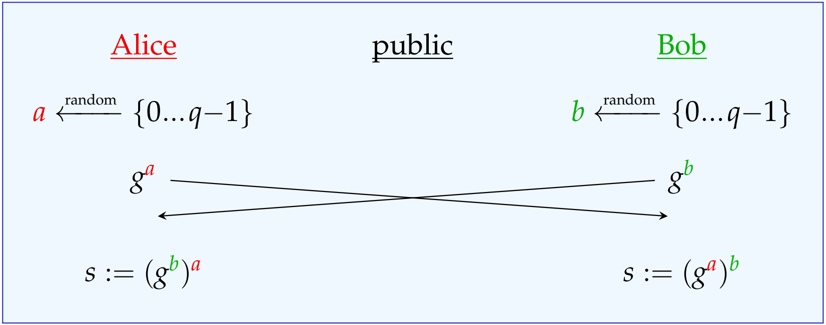

Fundamental reason this works: $\cdot^a$ and $\cdot^b$ are commutative!

Diffie–Hellman: Bob vs. Eve

<u>Bob</u>

1. Set $t \leftarrow g$.
2. Set $t \leftarrow t \cdot g$.
3. Set $t \leftarrow t \cdot g$.
4. Set $t \leftarrow t \cdot g$.

...

$b{-}2$. Set $t \leftarrow t \cdot g$.

$b{-}1$. Set $t \leftarrow t \cdot g$.

b. Publish $B \leftarrow t \cdot g$.

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Is this a good idea?

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Attacker Eve

1. Set $t \leftarrow g$. If $t = B$ return 1.
2. Set $t \leftarrow t \cdot g$. If $t = B$ return 2.
3. Set $t \leftarrow t \cdot g$. If $t = B$ return 3.
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$b+1$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b + 1$.

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Diffie–Hellman: Bob vs. Eve

Bob	Attacker Eve
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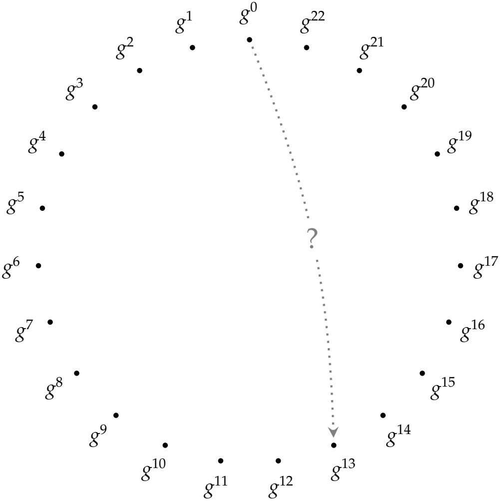
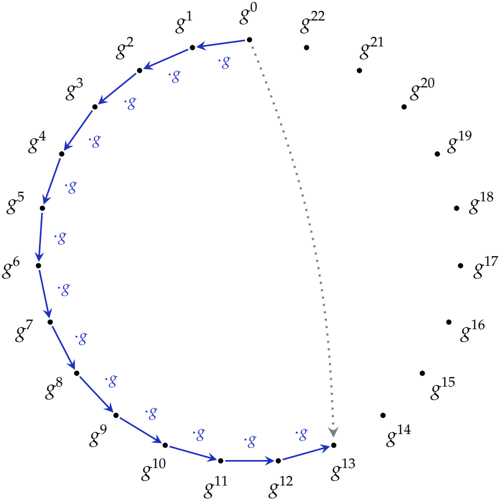
Effort for both: $O(\#G)$. Bob needs to be smarter.

(This attacker is also kind of dumb, but that doesn't matter for my point here.)



Bob computes his public key g^{13} from g.

multiply



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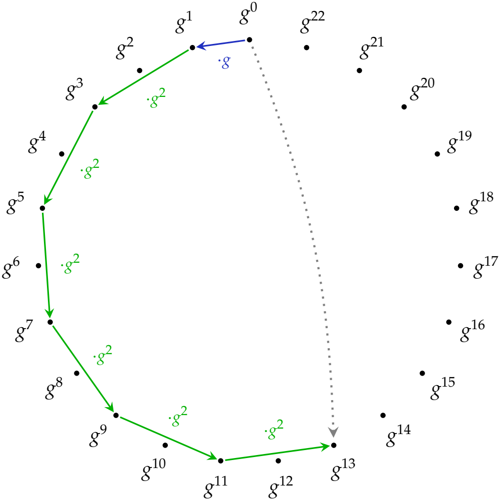
Square-and-multiply



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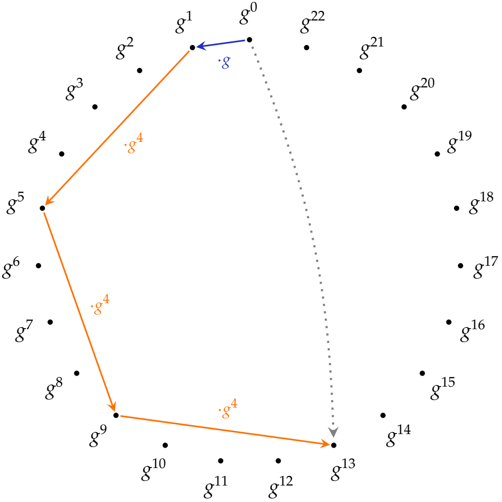
Square-and-multiply-and-square-and-multiply



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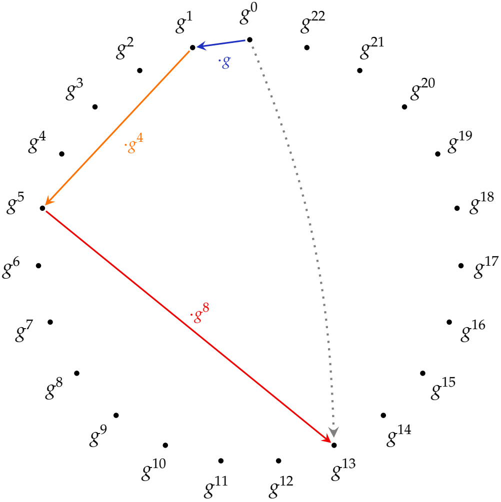

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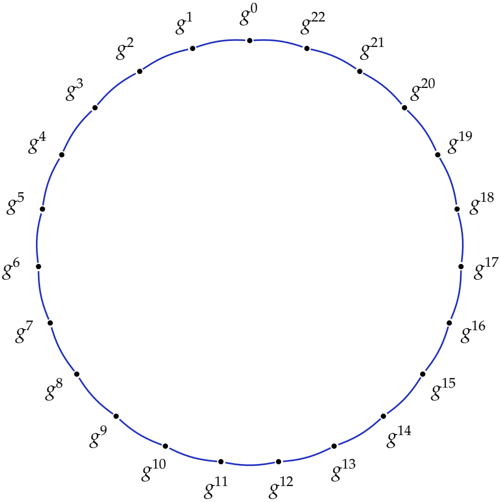
Square-and-multiply as graphs



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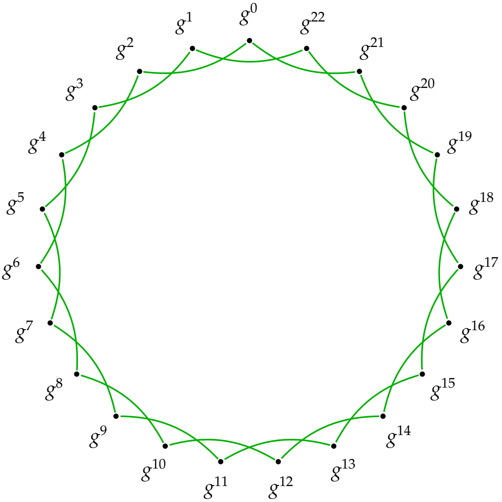

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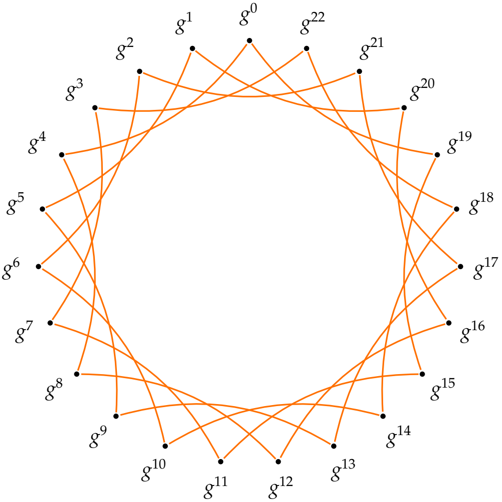

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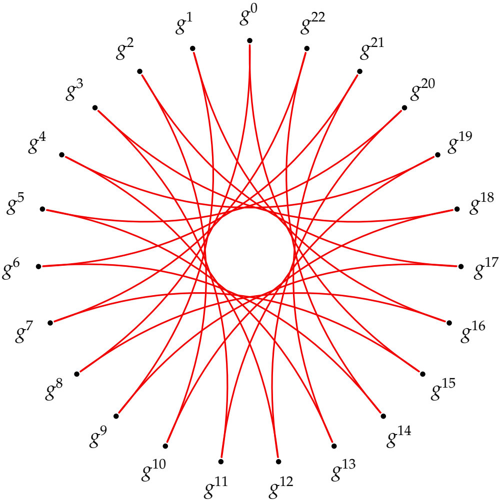
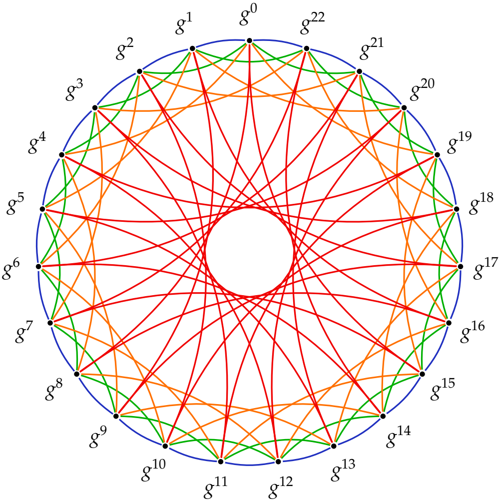


Square-and-multiply as a graph



Crypto on graphs?

We’ve been doing it all the time!

The fast mixing requirement

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Shor's quantum algorithm computes α from g^α in any group in polynomial time.

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isogeny graphs

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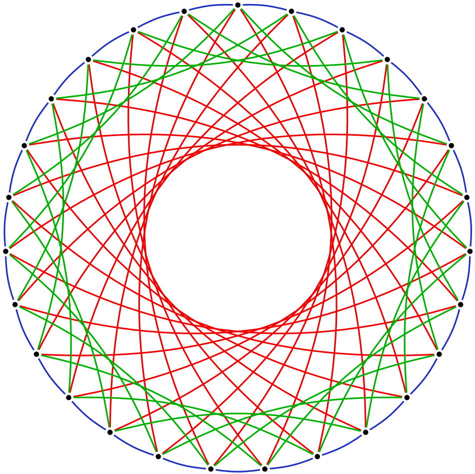
Components of particular isogeny graphs look like this:



Plan for this lecture

- High-level overview for intuition. ✓
- Recap: Elliptic curves & isogenies.
- The CSIDH non-interactive key exchange.
- Classical and quantum security of CSIDH.
- Orientations and the SCALLOP family.
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Reminder:

A rational function is $f(x, y)/g(x, y)$ where f, g are polynomials.

A group homomorphism φ satisfies $\varphi(P + Q) = \varphi(P) + \varphi(Q)$.

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The kernel of an isogeny $\varphi\colon E \to E'$ is $\{P \in E : \varphi(P) = \infty\}$.
The degree of a separable* isogeny is the size of its kernel.

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Example #1: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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Example #2: For each $m \neq 0$, the multiplication-by-m map

$$[m]\colon E \to E$$

is a degree-m^2 isogeny.

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Important fact: An isogeny φ is $\mathbb{F}_q$-rational iff $\pi \circ \varphi = \varphi \circ \pi$.

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sage: mu.rational_maps()
((x^25 + x^23 + ... + 14*x^3 + 25*x)
    /(25*x^24 + 14*x^22 - ... + x^2 + 1),
 (50*x^36*y + 20*x^34*y + ... + 45*x^2*y + 48*y)
    /(-12*x^36 - 2*x^34 + ... - 26*x^2 + 50))
```

Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable* isogeny $\varphi_G : E \to E'$ with kernel G.

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⇝ Decompose large-degree isogenies into prime steps.
That is: Walk in an isogeny graph.

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Elliptic Curve defined by y^2 = x^3 + x
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sage: phi = E.isogeny(K)
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Isogeny of degree 7
    from Elliptic Curve defined by y^2 = x^3 + x
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    to Elliptic Curve defined by y^2 = x^3 + 285*x + 87
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sage: phi.rational_maps()
((x^7 + 129*x^6 - ... + 25)/(x^6 + 129*x^5 - ... + 36),
 (x^9*y - 16*x^8*y - ... + 70*y)/(x^9 - 16*x^8 + ...))
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Consider a field k and let $S \not\ni \mathrm{char}(k)$ be a set of primes.

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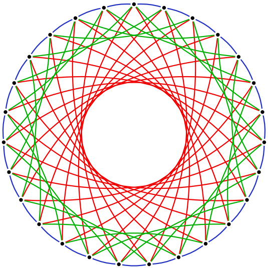
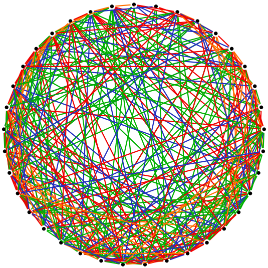


$k = \mathbb{F}_{419}, \ S = \{3, 5, 7\}$



$k = \mathbb{F}_{431^2}, \ S = \{2, 3, 5, 7\}$.

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Solution:

Let $p \geq 5$ be prime.

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- In that case, $E(\mathbb{F}_p) \cong \mathbb{Z}/(p{+}1)$ and
  $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}/(p{+}1) \times \mathbb{Z}/(p{+}1)$.

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(All curves are supersingular until about 14:00.)

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CSIDH [ˈsiːˌsaɪd]

[Castryck–Lange–Martindale–Panny–Renes 2018]

Isogeny-based key exchange: High-level view

E

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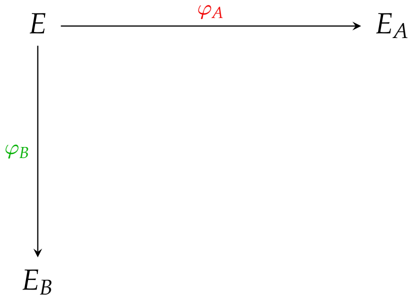

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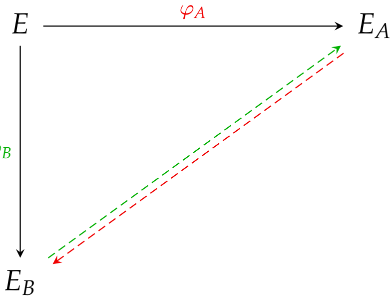

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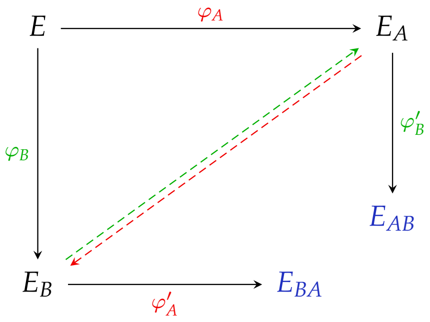
- Alice and Bob transmit the end curves E_A and E_B.

Isogeny-based key exchange: High-level view



- Alice & Bob pick secret $\varphi_A : E \to E_A$ and $\varphi_B : E \to E_B$.
  (These isogenies correspond to walking on the isogeny graph.)
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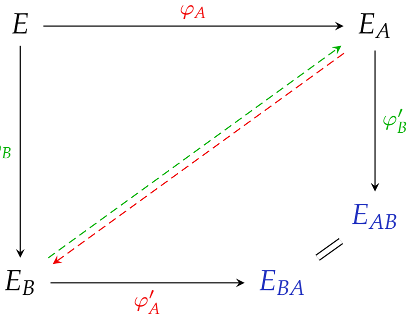

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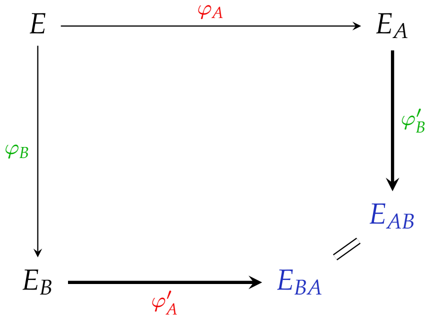
How to find "parallel" isogenies?



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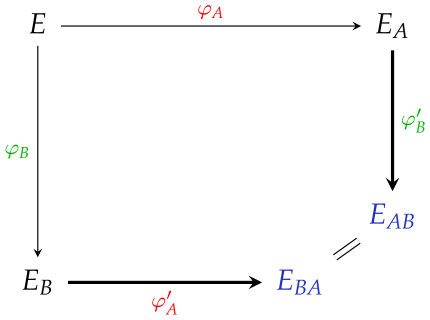

CSIDH’s solution (earlier: Couveignes, Rostovtsev–Stolbunov):

Use special isogenies φ_A which can be transported to the curve E_B totally independently of the secret isogeny φ_B.

(Similarly with reversed roles, of course.)

“Special” isogenies

Let $E/\mathbb{F}_p$ be supersingular and recall $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$.

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We consider prime ℓ and refer to φ_ℓ as a “special” isogeny.

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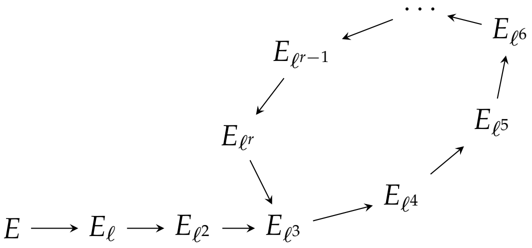

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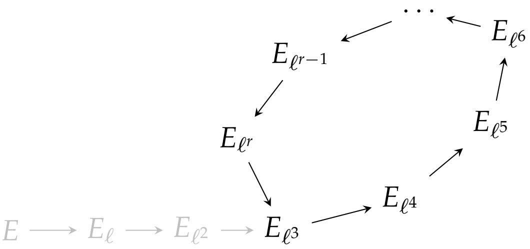

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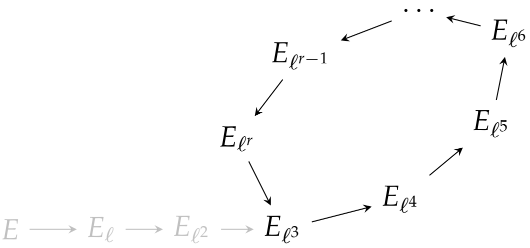

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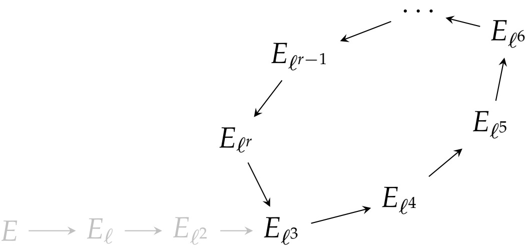

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$\Longrightarrow$ The "special" isogenies φ_ℓ form isogeny cycles!

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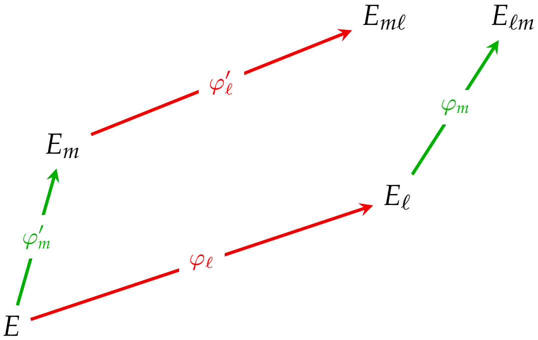

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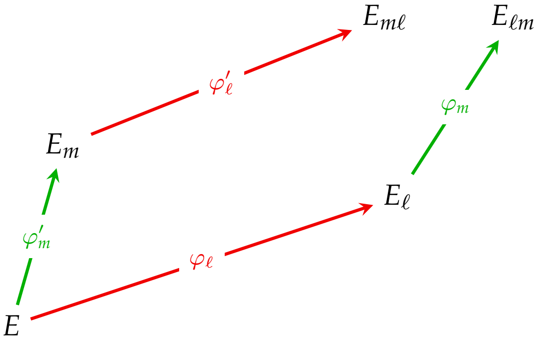

- Exercise: $\ker(\varphi'_\ell \circ \varphi'_m) = \ker(\varphi_m \circ \varphi_\ell) = \langle \ker \varphi_\ell, \ker \varphi'_m \rangle$.

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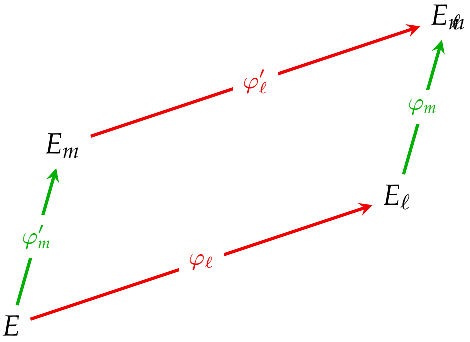

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!! The order cannot matter $\implies$ cycles must be compatible.

CSIDH in one slide

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- Choose some small odd primes $\ell_1, ..., \ell_n$.
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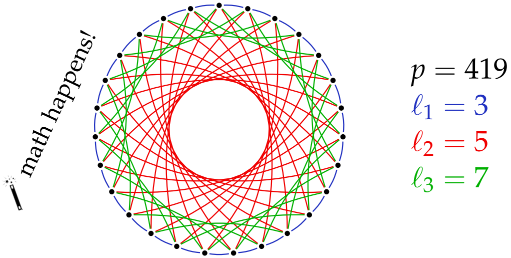

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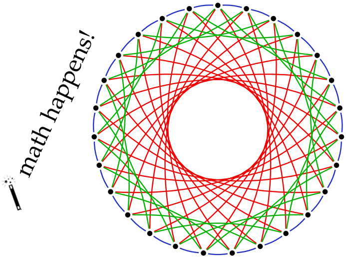


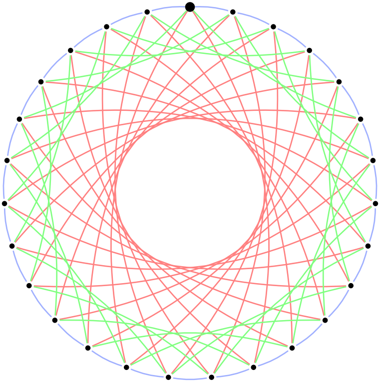
$p = 419$
$\ell_1 = 3$
$\ell_2 = 5$
$\ell_3 = 7$

- Walking "left" and "right" on any ℓ_i-subgraph is efficient.

CSIDH key exchange

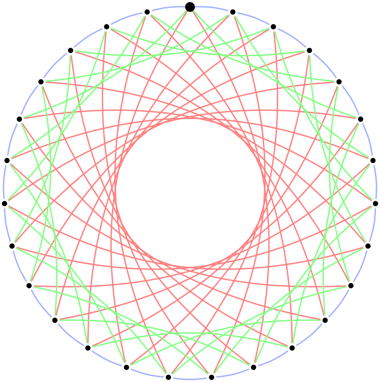
Alice

[+, +, −, −]



Bob

[−, +, −, −]



CSIDH key exchange

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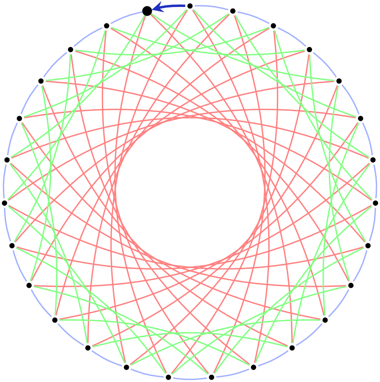
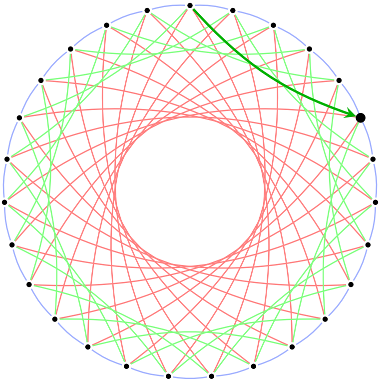
[+, +, −, −]
↑



Bob

[−, +, −, −]
↑



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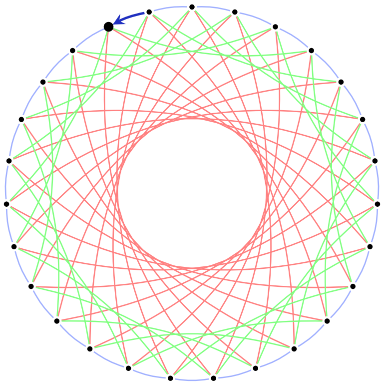

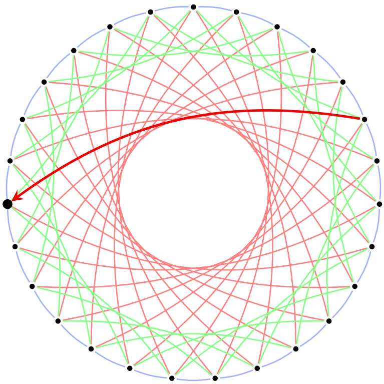

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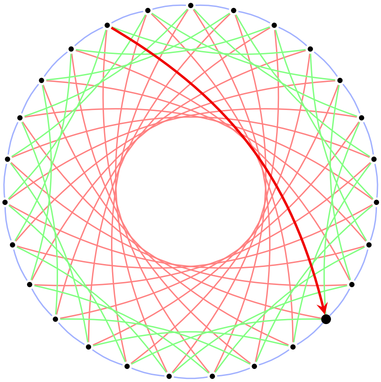

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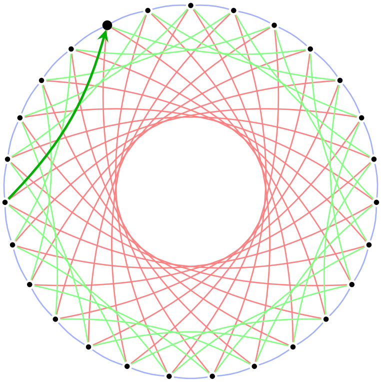

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↑

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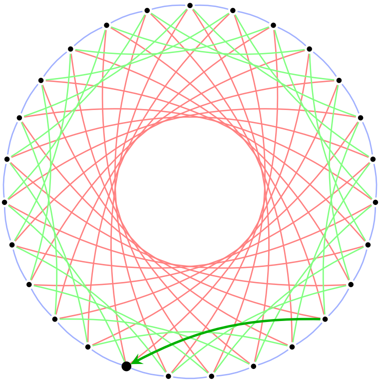

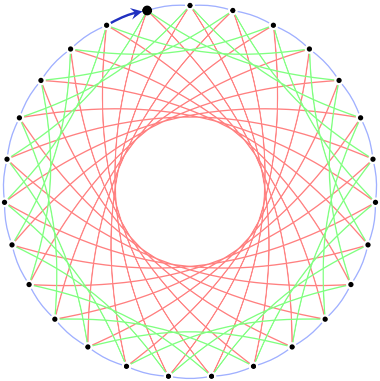

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[+, +, −, −]

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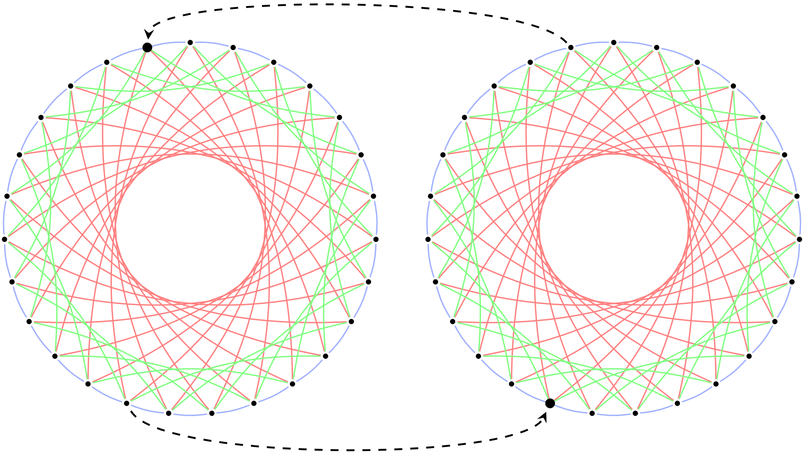

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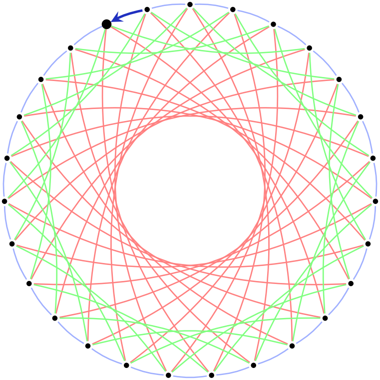

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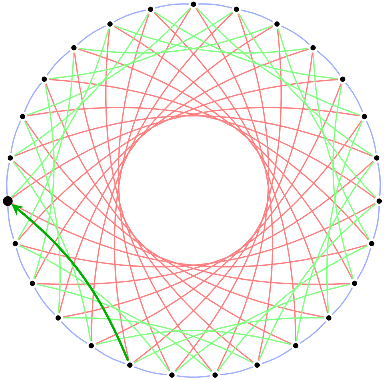

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[**+**, **+**, **−**, **−**]
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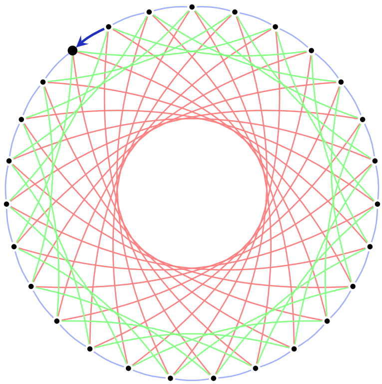

Bob

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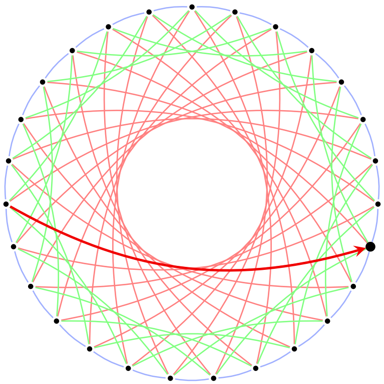

CSIDH key exchange

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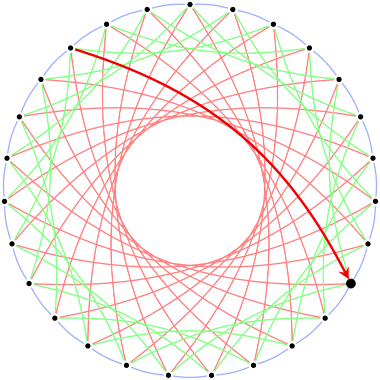

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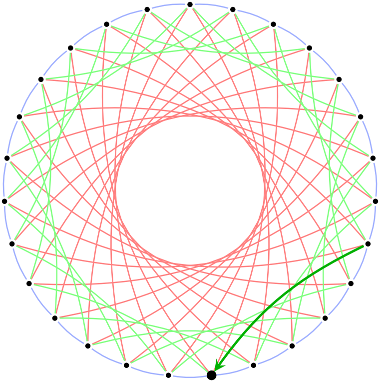

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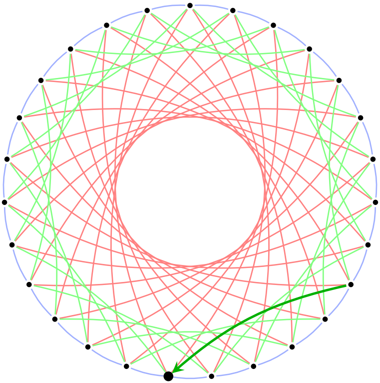

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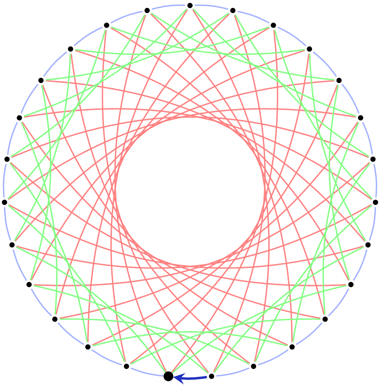

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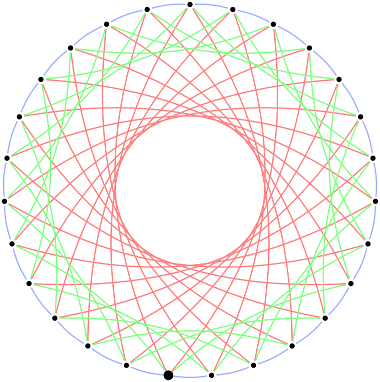

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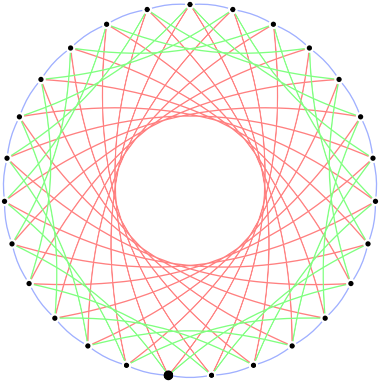

And... action! 🎬

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Example: [**+**, **+**, **−**, **−**, **−**, **+**, **−**, **−**] just becomes $(+1, \quad 0, -3) \in \mathbb{Z}^3$.

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There is a group action of $(\mathbb{Z}^n, +)$ on our set of curves X!

(An action of a group $(G, \cdot)$ on a set X is a map $*\colon G \times X \to X$
such that $id * x = x$ and $g * (h * x) = (g \cdot h) * x$ for all $g, h \in G$ and $x \in X$.)

The class group

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!! This group characterizes *when two paths lead to the same curve*.

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(Finding a point of order ℓ_i: Pick $x \in \mathbb{F}_p$ random. Find $y \in \mathbb{F}_{p^2}$ such that $P = (x, y) \in E$. Compute $Q = [\frac{p+1}{\ell_i}]P$. Hope that $Q \neq \infty$, else retry.)

In SageMath:

```
sage: E = EllipticCurve(GF(419^2), [1,0])
sage: E
Elliptic Curve defined by y^2 = x^3 + x
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....:     try:
....:         P = E.lift_x(x)
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sage: P
(218 : 403 : 1)
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....:
sage: P
(218 : 403 : 1)
sage: P.order().factor()
2 * 3 * 7
sage: EE = E.isogeny_codomain(2*3*P)  # "left" 7-step
sage: EE
Elliptic Curve defined by y^2 = x^3 + 285*x + 87
          over Finite Field in z2 of size 419^2
```

Efficient x-only arithmetic

- For $n \in \mathbb{Z}$, we have $[n](-P) = -[n]P$. (This holds in any group.)

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The same reasoning works for isogeny formulas.

Net result: With x-only arithmetic everything happens over $\mathbb{F}_p$.
$\Longrightarrow$ (Relatively) efficient CSIDH implementations!

Plan for this lecture

- High-level overview for intuition. ✓
- Recap: Elliptic curves & isogenies. ✓
- The CSIDH non-interactive key exchange. ✓
- Classical and quantum security of CSIDH.
- Orientations and the SCALLOP family.
- *Un*restricted effective group actions.

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For group <u>actions</u>, we simply cannot compose $a * s$ and $b * s$!

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Core problem:
Given $E, E' \in X$, find a path $E \to E'$ in the isogeny graph.

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$\rightsquigarrow$ best known classical attack: meet-in-the-middle, $\tilde{\mathcal{O}}(p^{1/4})$.

Fully exponential: Complexity $\exp\big((\log p)^{1+o(1)}\big)$.

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Solving abelian hidden shift breaks CSIDH.

$\rightsquigarrow$ non-devastating quantum attack (Kuperberg's algorithm).

Subexponential: Complexity $\exp\big((\log p)^{1/2+o(1)}\big)$.

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$\Longrightarrow$ Security estimates for CSIDH vary wildly.

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Fact: If $\varphi\colon E \to E'$ is an isogeny for which $\ker(\varphi)$ is described in terms of scalars and some endomorphism $\tau \in \mathrm{End}(E)$, then we can usually* push τ through φ:

$$\begin{aligned} \mathbb{Z}[\tau] &\hookrightarrow \mathrm{End}(E') \\ \tau &\longmapsto (\varphi \circ \tau \circ \widehat{\varphi})/\deg(\varphi) \end{aligned}$$

* Devils in details.

Ideals $\leftrightarrow$ kernels

More precisely, the subsets of endomorphisms which determine isogeny kernel subgroups are ideals of the endomorphism ring.

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$\rightsquigarrow$ Connection to the "class set" or class group:

ideals	$\longleftrightarrow$	kernels	$\longleftrightarrow$	isogenies
ideal *classes*	$\longleftrightarrow$	(no name)	$\longleftrightarrow$	isogeny *codomains*

Orientations & oriented curves

Let $\mathcal{O} = \mathbb{Z}[\tau]$ be an imaginary-quadratic order.

(Standard cases: $\tau = \sqrt{-d}$ or $\tau = \frac{1+\sqrt{-d}}{2}$ where $d \in \mathbb{Z}_{\geq 1}$.)

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Example: Any nonscalar endomorphism $\tau \in \mathrm{End}(E) \setminus \mathbb{Z}$ defines an orientation of $\mathcal{O} := \mathbb{Z}[\tau]$ on E.

The oriented class-group action

Onuki 2020 (previously Kohel–Colò without proof):

Theorem 3.4. *Let K be an imaginary quadratic field such that p does not split in K, and $\mathcal{O}$ an order in K such that p does not divide the conductor of $\mathcal{O}$. Then the ideal class group $\mathcal{C}\ell(\mathcal{O})$ acts freely and transitively on $\rho(\mathcal{E}\ell\ell(\mathcal{O}))$.*

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$\rho(\mathcal{E}\ell\ell(\mathcal{O}))$: a set of supersingular elliptic curves E over $\mathbb{F}_{p^2}$ with a primitive orientation $\iota\colon \mathcal{O} \hookrightarrow \mathrm{End}(E)$, up to oriented isomorphism.

- $\iota\colon \mathcal{O} \hookrightarrow \mathrm{End}(E)$ is *primitive* if $(\iota(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathrm{End}(E) = \iota(\mathcal{O})$.
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The group action is defined as follows:

$$\mathfrak{a} \star (E, \iota) := \left(E/\mathfrak{a},\ (\phi_{\mathfrak{a}} \circ \iota \circ \widehat{\phi}_{\mathfrak{a}})/\mathrm{norm}(\mathfrak{a})\right)$$

where $\phi_{\mathfrak{a}} : E \to E/\mathfrak{a}$ is the isogeny with kernel

$$E[\mathfrak{a}] := \bigcap_{\alpha \in \mathfrak{a}} \ker(\iota(\alpha)) .$$

Recap: CSIDH

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$$\mathfrak{l}_i := (\ell_i, \pi - 1)\,;$$
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where π is the p-power Frobenius endomorphism ($\pi^2 = [-p]$).

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Fun fact: Orienting $E/\mathbb{F}_p$ by $\sqrt{-p} \mapsto -\pi$ gives exactly the same picture, but everything is mirrored along "quadratic twisting":

$$\{y^2 = x^3 + Ax^2 + x\} \xrightarrow{\sim} \{y^2 = x^3 - Ax^2 + x\}$$

Representing orientations

To turn the previous theorem into a concrete group action for general $\mathcal{O}$, we need to specify how to encode the pair (E, ι):

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$\rightsquigarrow$ In practice, an oriented curve is given as a pair (E, ϑ) with $\vartheta \in \mathrm{End}(E)$, implicitly communicating that $\vartheta = \iota(\tau)$.

- There are multiple options for representing such a ϑ.
  Simple example: A deterministically chosen generator point of $\ker(\vartheta)$.
  More complicated: Deterministic "HD" representation (SCALLOP-HD).

Oriented isogeny group actions: Why?

- Key point: Orientations allow us to decouple the discriminant of $\mathcal{O}$ from the characteristic p.

This is advantageous for at least two reasons (see next part):

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$\rightsquigarrow$ For Clapoti, we have to solve norm equations that are derived from $\mathcal{O}$ for target values derived from p.

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The basic strategy à la C/R–S

- Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ be small prime ideals of $\mathcal{O}$, and suppose $\mathfrak{a}$ is given to us in the form $\mathfrak{a} = \mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_n^{e_n}$.
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- Optimizations: Batch multiple $\mathfrak{l}_i$ together $\rightsquigarrow$ "strategies".

The basic problem with the basic strategy

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$\rightsquigarrow$ A priori **not** an effective group action when done either way!

The CSI-FiSh approach

...combines exponent vectors with reduction by exploiting the relation lattice of the chosen ideal classes. It works as follows:

The strategy to act by a given, arbitrarily long and ugly exponent vector $\underline{v} \in \mathbb{Z}^d$ consists of the following steps:

1. "Computing the class group": Find a basis of the *relation lattice* $\Lambda \subseteq \mathbb{Z}^d$ with respect to $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$.
   [Classically subexponential-time, quantumly polynomial-time. Precomputation.]
2. "Lattice reduction": Prepare a "good" basis of Λ using a lattice-reduction algorithm such as BKZ.
   [Configurable complexity-quality tradeoff by varying the block size. Precomputation.]
3. "Approximate CVP": Obtain a vector $\underline{w} \in \Lambda$ such that $\|\underline{v} - \underline{w}\|_1$ is "small", using the reduced basis.
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https://yx7.cc/blah/2023-04-14.html

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What about asymptotics?

Tradeoff: Lattice part vs. isogeny part

- By increasing the number n of ideals $\mathfrak{l}_i$, we can trade off some "isogeny effort" for "lattice effort".

$\rightsquigarrow$ Sweet spot: Minimize total cost.

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CSI-FiSh really isn't polynomial-time

It is fairly well-known that CSIDH[1] in its basic form is merely a *restricted* effective group action $G \times X \to X$: There is a small number of group elements $\mathfrak{l}_1, \ldots, \mathfrak{l}_d \in G$ whose action can be applied to arbitrary elements of X efficiently, but applying other elements (say, large products $\mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_d^{e_d}$ of the $\mathfrak{l}_i$) quickly becomes infeasible as the exponents grow.

The only known method to circumvent this issue consists of a folklore strategy first employed in practice by the signature scheme CSI-FiSh. The core of the technique is to rewrite any given group element as a *short* product combination of the $\mathfrak{l}_i$, whose action can then be computed in the usual way much more affordably. (Notice how this is philosophically similar to the role of the square-and-multiply algorithm in discrete-logarithm land!)

The main point of this post is to remark that this approach is **not asymptotically efficient**, even when a quantum computer can be used, contradicting a false belief that appears to be rather common among isogeny aficionados.

$\rightsquigarrow$

- Classically: Evaluation $L_p[1/2]$. Attack $L_p[1]$.
- Quantumly: Evaluation $L_p[1/3]$. Attack $L_p[1/2]$.

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Clapoti

Even more maritime isogenies??

Noun [edit]

clapotis *m* (*plural* **clapotis**)

1. lapping of water against a surface [synonyms ▲]

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- Page–Robert: A polynomial-time algorithm to evaluate the isogeny group action on arbitrary ideals.

Polynomial-time group action: Clapoti

<u>Idea:</u>

- Find two ideals $\mathfrak{b}$, $\mathfrak{c}$ of coprime norms, both equivalent to $\mathfrak{a}$.
  Let $N := \mathrm{norm}(\mathfrak{b}) + \mathrm{norm}(\mathfrak{c})$.

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$$\begin{array}{ccc} E & \xrightarrow{\phi_{\mathfrak{b}}} & E_{\mathfrak{a}} \\ {\scriptstyle \phi_{\overline{\mathfrak{c}}}}\downarrow & & \downarrow{\scriptstyle \psi_{\overline{\mathfrak{c}}}} \\ E_{\overline{\mathfrak{a}}} & \xrightarrow[\psi_{\mathfrak{b}}]{} & E \end{array}$$

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\end{array}
$$

- Kani: This gives an N-isogeny

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\begin{aligned}
\Phi \colon E \times E &\longrightarrow E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}}, \\
(P, Q) &\longmapsto \left(\phi_{\mathfrak{b}}(P) + \widehat{\psi_{\overline{\mathfrak{c}}}}(Q),\ -\phi_{\overline{\mathfrak{c}}}(P) + \widehat{\psi_{\mathfrak{b}}}(Q)\right).
\end{aligned}
$$

Polynomial-time group action: Clapoti

<u>Idea:</u>

- Find two ideals $\mathfrak{b}, \mathfrak{c}$ of coprime norms, both equivalent to $\mathfrak{a}$.
  Let $N := \mathrm{norm}(\mathfrak{b}) + \mathrm{norm}(\mathfrak{c})$.

$$\begin{array}{ccc} E & \xrightarrow{\phi_{\mathfrak{b}}} & E_{\mathfrak{a}} \\ {\scriptstyle \phi_{\overline{\mathfrak{c}}}}\downarrow & & \downarrow{\scriptstyle \psi_{\overline{\mathfrak{c}}}} \\ E_{\overline{\mathfrak{a}}} & \xrightarrow[\psi_{\mathfrak{b}}]{} & E \end{array}$$

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- The kernel is $\ker(\Phi) = \left\{\left(\widehat{\phi}_{\mathfrak{b}}(R), \psi_{\overline{\mathfrak{c}}}(R)\right) : R \in E_{\mathfrak{a}}[N]\right\}$.

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The kernel is equal to the alternative description

$$\ker(\Phi) = \left\{ \left([\mathrm{norm}(\mathfrak{b})]R, \gamma(R) \right) \mid R \in E[N] \right\}$$

where $\gamma \in \mathrm{End}(E)$ is a generator of the principal ideal $\mathfrak{b}\overline{\mathfrak{c}}$.

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$\Longrightarrow$ The isogeny group action can now be computed in polynomial time even for "ugly" input ideals.

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$\Longrightarrow$ The isogeny group action can now be computed in polynomial time even for "ugly" input ideals.

$\Longrightarrow$ Isogenies yield true effective group actions, at last!

Plan for this lecture

- High-level overview for intuition. ✓
- Recap: Elliptic curves & isogenies. ✓
- The CSIDH non-interactive key exchange. ✓
- Classical and quantum security of CSIDH. ✓
- Orientations and the SCALLOP family. ✓
- *Un*restricted effective group actions. ✓

Questions?

(Also feel free to email me: `lorenz@yx7.cc`)