30−ε Years of Isogeny Group Actions

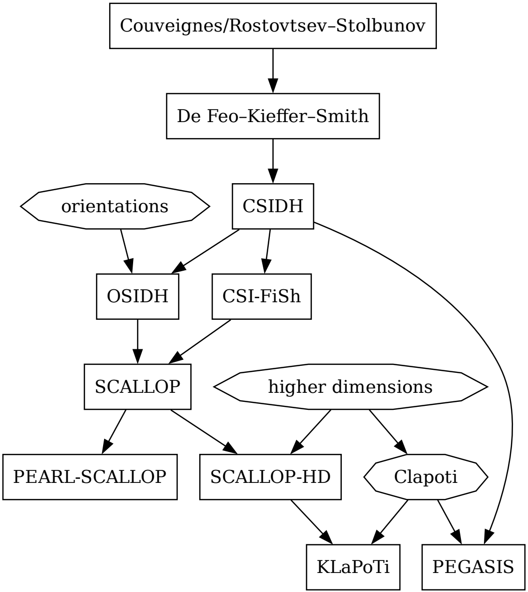
Lorenz Panny

Technische Universität München

Swissogeny Day, Zürich, 20 March 2025

Plan for this talk

- The CSIDH non-interactive key exchange.
- Classical and quantum security.
- Is this an effective group action?
- Oriented elliptic curves and isogenies.
- *Un*restricted effective group actions.


Couveignes/Rostovtsev–Stolbunov
De Feo–Kieffer–Smith
orientations
CSIDH
OSIDH
CSI-FiSh
SCALLOP
higher dimensions
PEARL-SCALLOP
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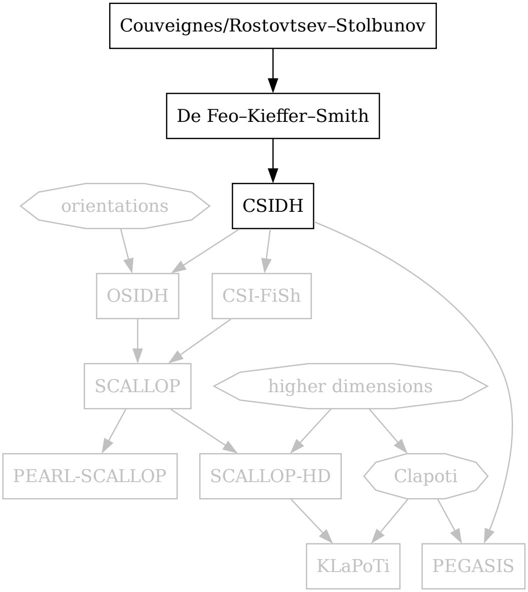
PEGASIS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CSIDH [ˈsiːˌsaɪd]

[Castryck–Lange–Martindale–Panny–Renes 2018]

Isogeny-based key exchange: High-level view

E

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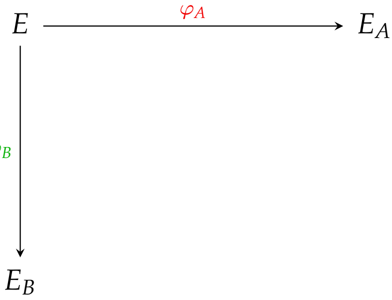

- Alice & Bob pick secret $\varphi_A \colon E \to E_A$ and $\varphi_B \colon E \to E_B$.
  (These isogenies correspond to walking on the isogeny graph.)

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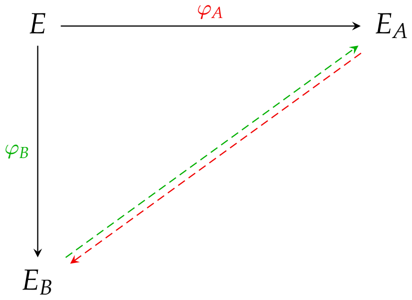

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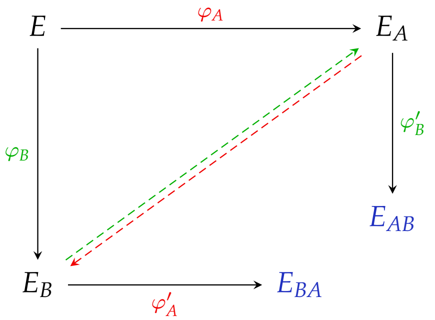

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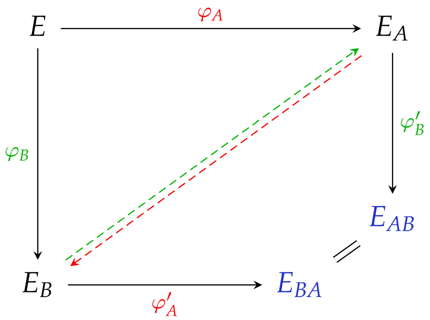

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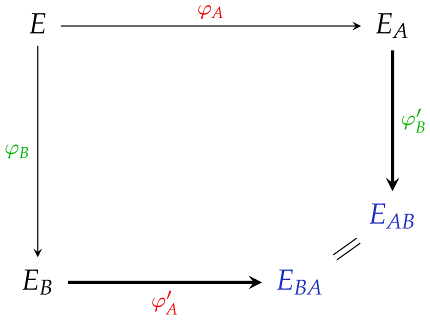
- Alice and Bob transmit the end curves E_A and E_B.
- Alice <u>somehow</u> finds a "parallel" $\varphi_{A'} : E_B \to E_{BA}$, and
  Bob <u>somehow</u> finds $\varphi_{B'} : E_A \to E_{AB}$, such that $E_{AB} \cong E_{BA}$.

How to find “parallel” isogenies?



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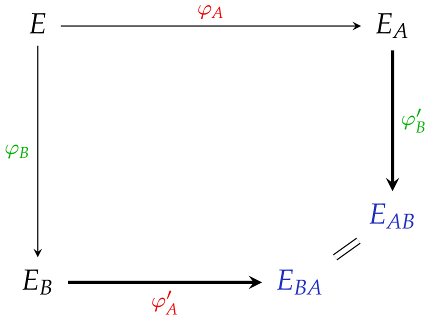

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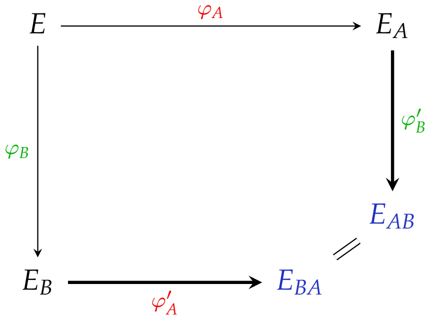

CSIDH's solution (earlier: Couveignes, Rostovtsev–Stolbunov):
Use special isogenies φ_A which can be transported to the curve E_B totally independently of the secret isogeny φ_B.
(Similarly with reversed roles, of course.)

CSIDH in one slide

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- Choose some small odd primes $\ell_1, ..., \ell_n$.
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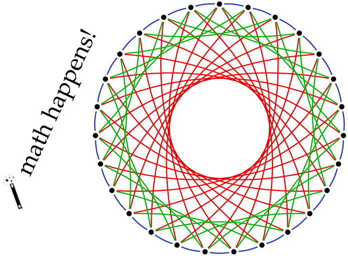

$p = 419$
$\ell_1 = 3$
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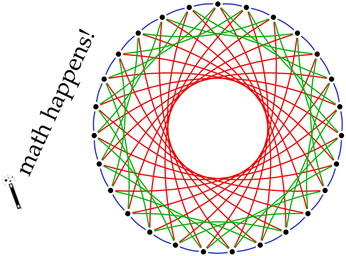

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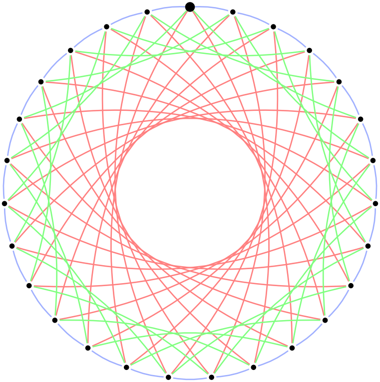
- Walking “left” and “right” on any ℓ_i-subgraph is efficient.

CSIDH key exchange

Alice

[+, +, −, −]



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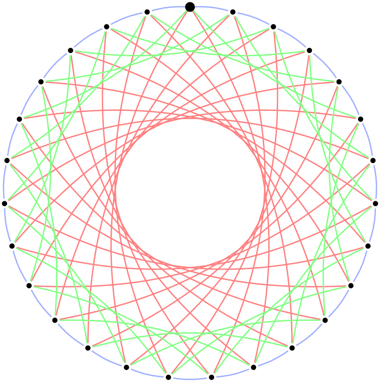
[−, +, −, −]



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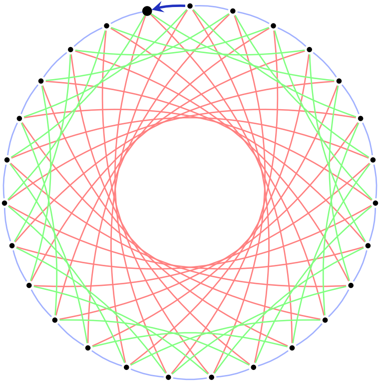

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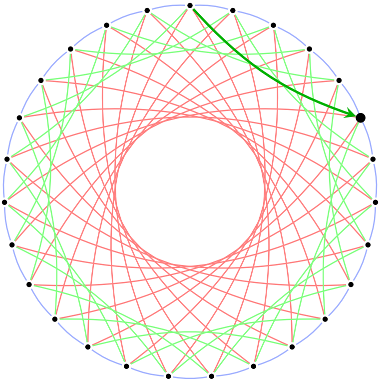

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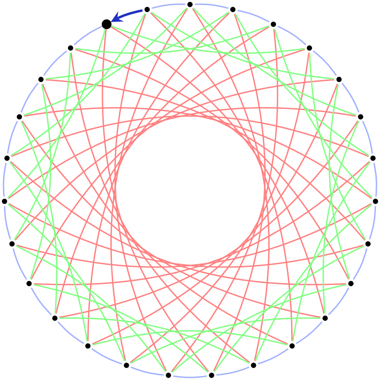

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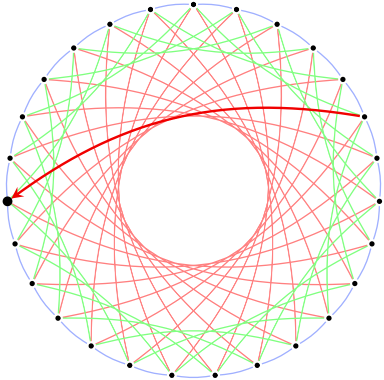

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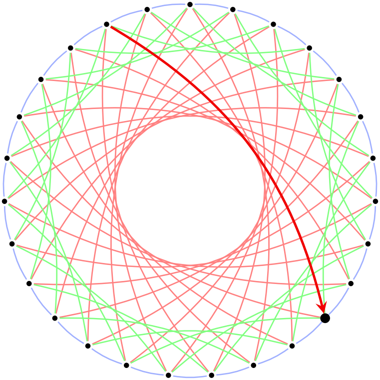

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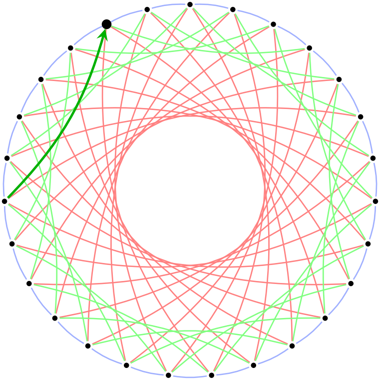

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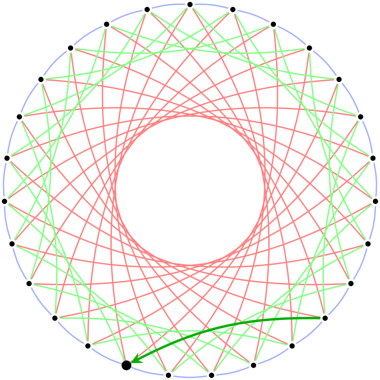

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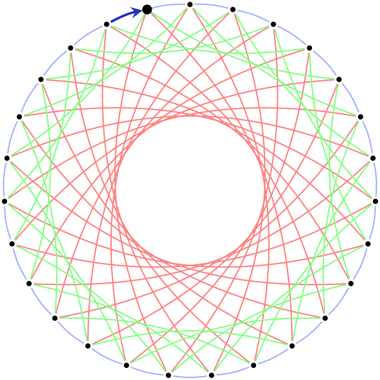

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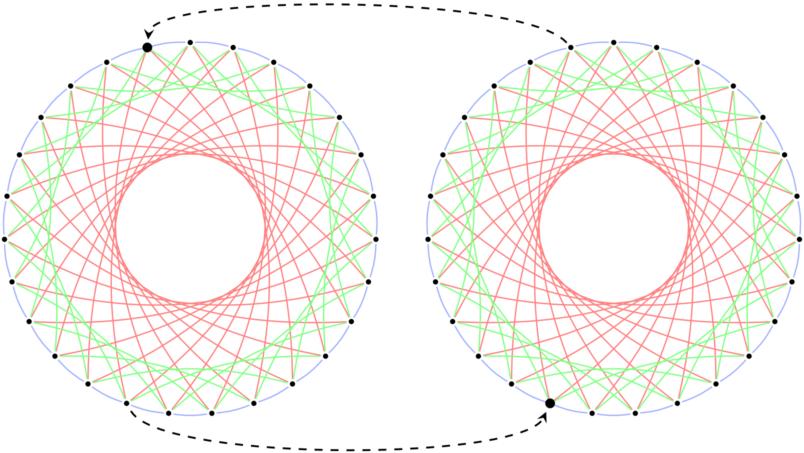

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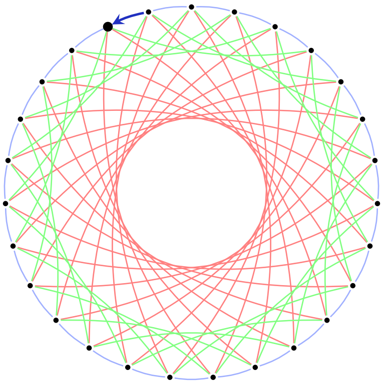

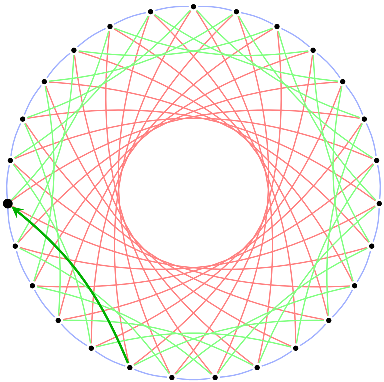

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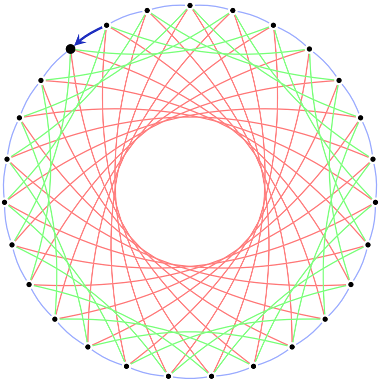

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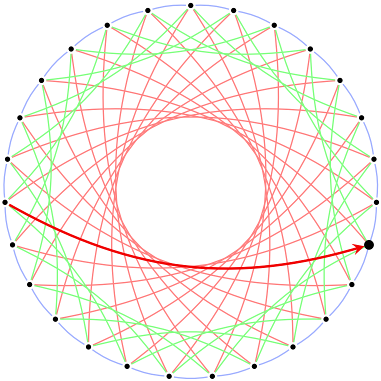

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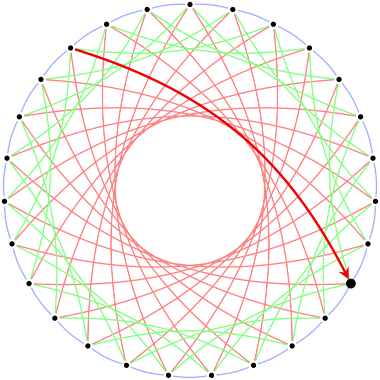

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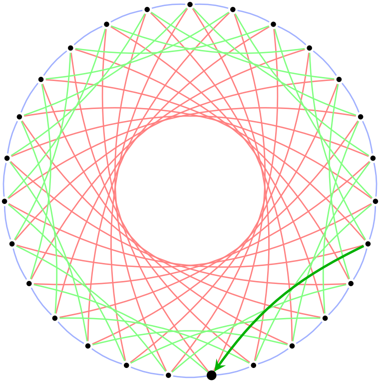

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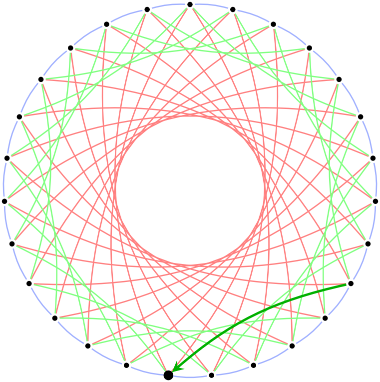

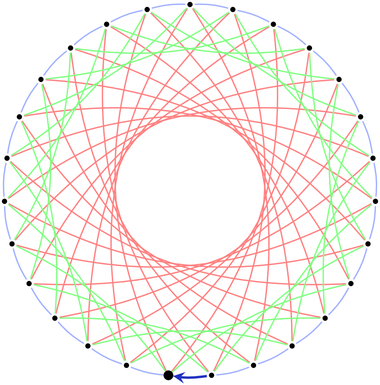

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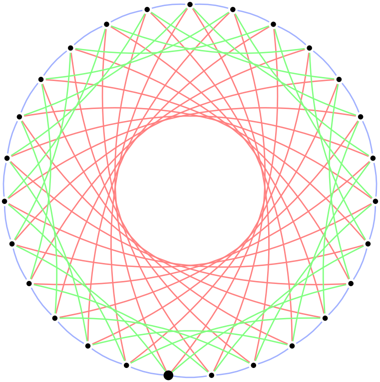

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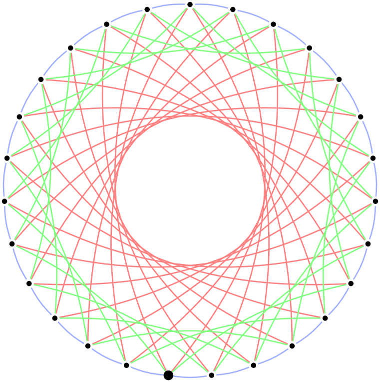

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There is a group action of $(\mathbb{Z}^n, +)$ on our set of curves X!

CSIDH via ideals

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$$\mathfrak{l}_i := (\ell_i, \pi - 1)\,;$$
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<u>General picture</u>: The kernels K of rational ℓ_i-isogenies are defined by ideals $\mathfrak{a}$ of $\mathrm{End}_{\mathbb{F}_p}(E)$ via

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!! The endomorphisms in $\mathfrak{a}$ “carve out” our kernel subgroup.

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!! This group characterizes *when two paths lead to the same curve*.

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$\rightsquigarrow$ Computing the action of $\mathfrak{l}_i$ is much more expensive.

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- The CSIDH non-interactive key exchange. ✓
- Classical and quantum security.
- Is this an effective group action?
- Oriented elliptic curves and isogenies.
- *Un*restricted effective group actions.

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For group <u>actions</u>, we simply cannot compose $a * s$ and $b * s$!

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Solving abelian hidden shift breaks CSIDH.

$\rightsquigarrow$ non-devastating quantum attack (Kuperberg's algorithm).

Subexponential: Complexity $\exp\big((\log p)^{1/2+o(1)}\big)$.

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  - $\rightsquigarrow$ **How to compare costs?**
    (Is one qubit operation $\approx$ one bit operation? a hundred? millions?)

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- The algorithm admits many different tradeoffs.
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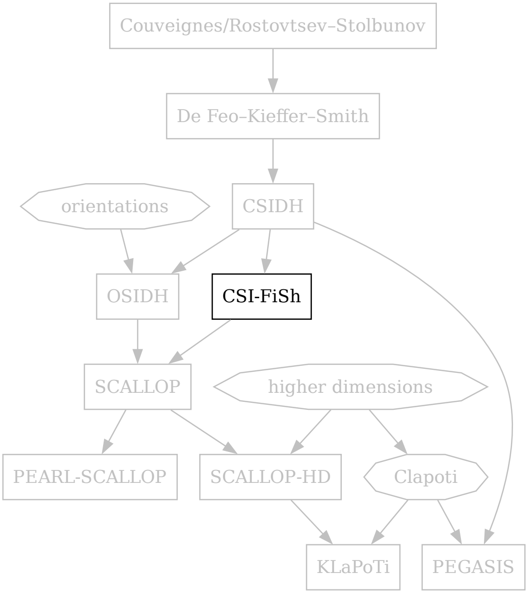
- The sieving phase has classical *and* quantum operations.
  - $\rightsquigarrow$ **How to compare costs?**
    (Is one qubit operation $\approx$ one bit operation? a hundred? millions?)

$\implies$ Security estimates for CSIDH & friends vary wildly.

Plan for this talk

- The CSIDH non-interactive key exchange. ✓
- Classical and quantum security. ✓
- Is this an effective group action?
- Oriented elliptic curves and isogenies.
- *Un*restricted effective group actions.


Couveignes/Rostovtsev–Stolbunov
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orientations
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The basic strategy à la C/RS/DKS/CSIDH

- Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ be small prime ideals of $\mathcal{O}$, and suppose $\mathfrak{a}$ is given to us in the form $\mathfrak{a} = \mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_n^{e_n}$.

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  Then the kernel is an order-ℓ_i point P with $\vartheta(P) = [\lambda_i]P$.
- Optimization: Batch multiple $\mathfrak{l}_i$ together $\rightsquigarrow$ "strategies".

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$\rightsquigarrow$ A priori **not** an effective group action when done either way!

The CSI-FiSh approach

...combines exponent vectors with reduction by exploiting the relation lattice of the chosen ideal classes. It works as follows:

The strategy to act by a given, arbitrarily long and ugly exponent vector $\underline{v} \in \mathbb{Z}^d$ consists of the following steps:

1. "Computing the class group": Find a basis of the *relation lattice* $\Lambda \subseteq \mathbb{Z}^d$ with respect to $\mathfrak{l}_1, \ldots, \mathfrak{l}_d$.
   [Classically subexponential-time, quantumly polynomial-time. Precomputation.]
2. "Lattice reduction": Prepare a "good" basis of Λ using a lattice-reduction algorithm such as BKZ.
   [Configurable complexity-quality tradeoff by varying the block size. Precomputation.]
3. "Approximate CVP": Obtain a vector $\underline{w} \in \Lambda$ such that $\|\underline{v} - \underline{w}\|_1$ is "small", using the reduced basis.
   [Polynomial-time, but the quality depends on the quality of step 2.]
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What about asymptotics?

Tradeoff: Lattice part vs. isogeny part

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CSI-FiSh really isn't polynomial-time

It is fairly well-known that CSIDH[1] in its basic form is merely a *restricted* effective group action $G \times X \to X$: There is a small number of group elements $\mathfrak{l}_1, \ldots, \mathfrak{l}_d \in G$ whose action can be applied to arbitrary elements of X efficiently, but applying other elements (say, large products $\mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_d^{e_d}$ of the $\mathfrak{l}_i$) quickly becomes infeasible as the exponents grow.

The only known method to circumvent this issue consists of a folklore strategy first employed in practice by the signature scheme CSI-FiSh. The core of the technique is to rewrite any given group element as a *short* product combination of the $\mathfrak{l}_i$, whose action can then be computed in the usual way much more affordably. (Notice how this is philosophically similar to the role of the square-and-multiply algorithm in discrete-logarithm land!)

The main point of this post is to remark that this approach is **not asymptotically efficient**, even when a quantum computer can be used, contradicting a false belief that appears to be rather common among isogeny aficionados.

$\rightsquigarrow$

- Classically: Evaluation $L_p[1/2]$. Attack $L_p[1]$.
- Quantumly: Evaluation $L_p[1/3]$. Attack $L_p[1/2]$.

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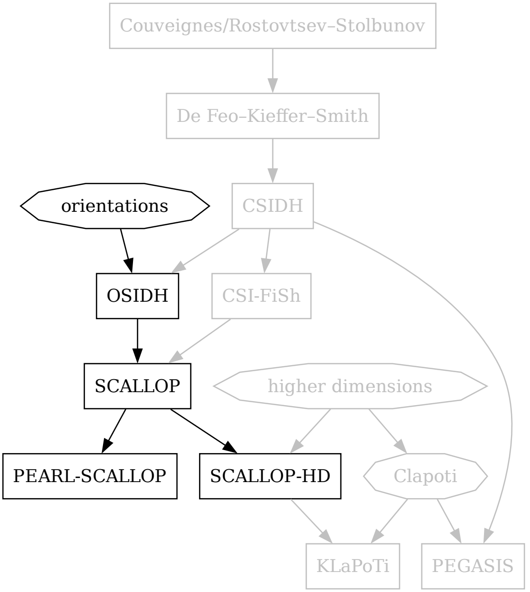
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Fact: If $\varphi\colon E \to E'$ is an isogeny for which $\ker(\varphi)$ is described in terms of scalars and some endomorphism $\tau \in \mathrm{End}(E)$, then we can usually push τ through φ:

$$\begin{aligned} \mathbb{Z}[\tau] &\longhookrightarrow \mathrm{End}(E') \\ \tau &\longmapsto (\varphi \circ \tau \circ \widehat{\varphi})/\deg(\varphi) \end{aligned}$$

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$\rightsquigarrow$ Connection to the "class set" or class group:

ideals	$\longleftrightarrow$	kernels	$\longleftrightarrow$	isogenies
ideal *classes*	$\longleftrightarrow$	(no name)	$\longleftrightarrow$	isogeny *codomains*

Orientations & oriented curves

Let $\mathcal{O} = \mathbb{Z}[\tau]$ be an imaginary-quadratic order.

(Standard cases: $\tau = \sqrt{-d}$ or $\tau = \frac{1+\sqrt{-d}}{2}$ where $d \in \mathbb{Z}_{\geq 1}$.)

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Example: Any nonscalar endomorphism $\tau \in \mathrm{End}(E) \setminus \mathbb{Z}$ defines an orientation of $\mathcal{O} := \mathbb{Z}[\tau]$ on E.

The oriented class-group action

Onuki 2020 (previously Kohel–Colò 2020 without proof):

Theorem 3.4. *Let K be an imaginary quadratic field such that p does not split in K, and $\mathcal{O}$ an order in K such that p does not divide the conductor of $\mathcal{O}$. Then the ideal class group $\mathcal{C}\ell(\mathcal{O})$ acts freely and transitively on $\rho(\mathcal{E}\ell\ell(\mathcal{O}))$.*

https://arxiv.org/pdf/2002.09894

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The group action is defined as follows:

$$\mathfrak{a} \star (E, \iota) := \left(E/\mathfrak{a},\ (\phi_{\mathfrak{a}} \circ \iota \circ \widehat{\phi}_{\mathfrak{a}})/\mathrm{norm}(\mathfrak{a})\right)$$

where $\phi_{\mathfrak{a}}\colon E \to E/\mathfrak{a}$ is the isogeny with kernel

$$E[\mathfrak{a}] := \bigcap_{\alpha \in \mathfrak{a}} \ker(\iota(\alpha))\,.$$

(NB: In the cases we care about, we have $\pi_{p^2} = [-p]$, hence all isogenies are $\mathbb{F}_{p^2}$-rational.)

Recap: CSIDH

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it need not be transmitted and we get an action on curves only.

Fun fact: Orienting $E/\mathbb{F}_p$ by $\sqrt{-p} \mapsto -\pi$ gives exactly the same picture, but everything is mirrored via quadratic twisting:

$$\{y^2 = x^3 + Ax^2 + x\} \overset{\sim}{\longmapsto} \{y^2 = x^3 - Ax^2 + x\}$$

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⇝ In **practice**, an oriented curve is given as a pair (E, ϑ) with $\vartheta \in \mathrm{End}(E)$, implicitly communicating that $\vartheta = \iota(\tau)$.

- There are multiple options for representing such a ϑ.
  Simple example: A deterministically chosen generator point of $\ker(\vartheta)$.
  More complicated: Deterministic HD representation (SCALLOP-HD).

Oriented isogeny group actions: Why?

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⇝ For Clapoti (soon!), we have to solve norm equations that are derived from $\mathcal{O}$ for target values derived from p.

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To play around with this, try my CTF challenge "`not_csidh`": `https://hxp.io/blog/96`
(Don't forget to submit your code to SageMath afterwards. :-))

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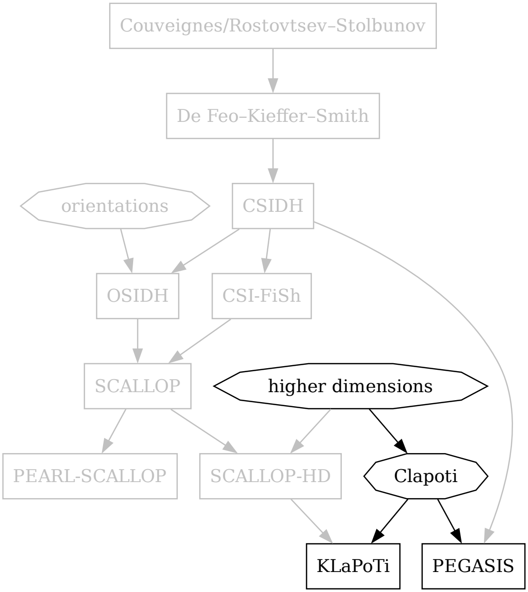
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CSI-FiSh
SCALLOP
higher dimensions
PEARL-SCALLOP
SCALLOP-HD
Clapoti
KLaPoTi
PEGASIS

Clapoti

Even more maritime isogenies??

Noun [edit]

clapotis *m* (*plural* **clapotis**)

1. lapping of water against a surface [synonyms ▲]

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- Page–Robert: A polynomial-time algorithm to evaluate the isogeny group action on arbitrary ideals.

Polynomial-time group action: Clapoti

Idea:

- Find two ideals $\mathfrak{b}$, $\mathfrak{c}$ of coprime norms, both equivalent to $\mathfrak{a}$.
  Let $N := \mathrm{norm}(\mathfrak{b}) + \mathrm{norm}(\mathfrak{c})$.

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- Kani: This gives an N-isogeny

$$\begin{aligned} \Phi\colon E \times E &\longrightarrow E_{\mathfrak{a}} \times E_{\bar{\mathfrak{a}}}, \\ (P, Q) &\longmapsto \left(\phi_{\mathfrak{b}}(P) + \widehat{\psi}_{\bar{\mathfrak{c}}}(Q),\ -\phi_{\bar{\mathfrak{c}}}(P) + \widehat{\psi}_{\mathfrak{b}}(Q)\right). \end{aligned}$$

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$\Longrightarrow$ Isogenies yield true effective group actions, at last!

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!! This is the norm form of $\mathcal{I} := \mathfrak{a} + \mathrm{i}\mathfrak{a}$ inside the quaternion order $\mathcal{Q} := \mathcal{O} + \mathrm{i}\mathcal{O}$. (NB: The quaternion algebra here is *not* $\mathrm{End}(E) \otimes \mathbb{Q}$.)

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⇝ KLaPoTi/SCALLOP2D: Practical instantiation of this.
Pretty bad performance for "small" parameters, but finally asymptotically polynomial-time for the first time.

Effective group actions (4D)

Applying Clapoti in $4 > 2$ dimensions is better.

$\rightsquigarrow$ **PEGASIS** $\leftrightsquigarrow$

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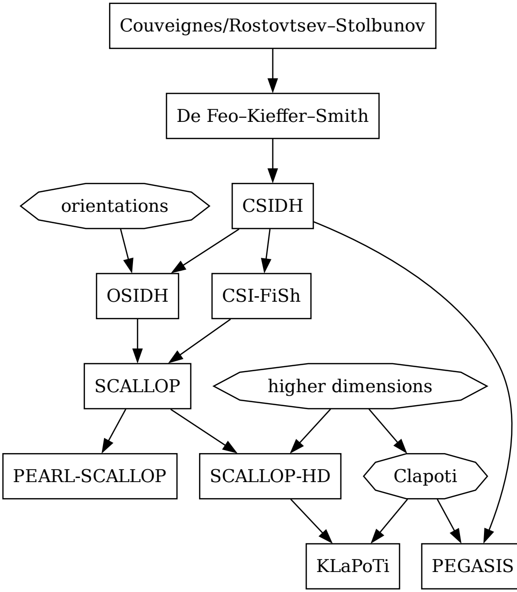
(See Ryan's talk later today!)


Couveignes/Rostovtsev–Stolbunov
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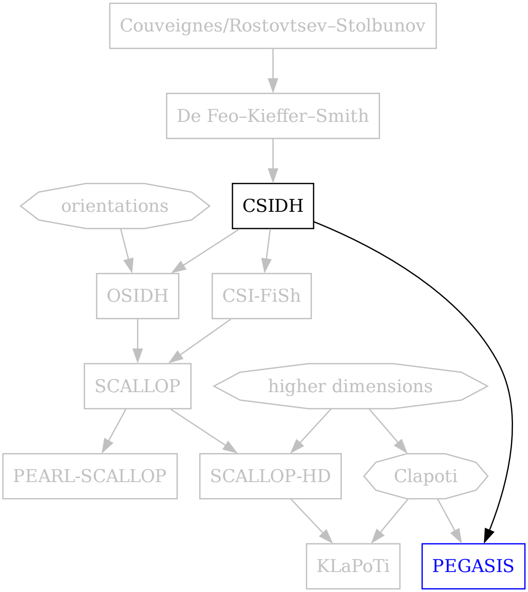
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Plan for this talk

- The CSIDH non-interactive key exchange. ✓
- Classical and quantum security. ✓
- Is this an effective group action? ✓
- Oriented elliptic curves and isogenies. ✓
- *Un*restricted effective group actions. ✓

Questions!

- Quantum security: How large should $\mathrm{disc}(\mathcal{O})$ be?
  (I think this is the biggest roadblock for CSIDH & friends.)
- Performance: Is PEGASIS universally superior?
  (Can we thank the others for their service and ditch them for good?)
- Protocols: Beyond key exchange?
  (Proposals exist — any of them convincing to practicioners?)

Questions?

(Also feel free to email me: `lorenz@yx7.cc`)