The state of the isogeny

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Workshop on the mathematics of post-quantum cryptography,
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Big picture

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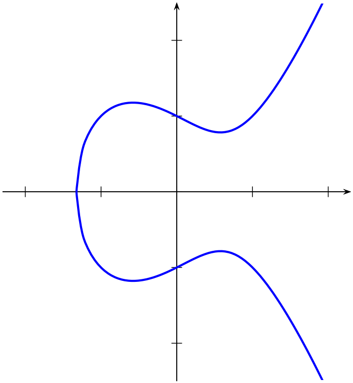
$\rightsquigarrow$ ***Cryptography!***

(Modern isogeny-based cryptography uses not just elliptic curves, but also higher-dimensional abelian varieties.)

Plan for this talk

- Elliptic curves & isogenies.
- The SIKE attacks.
- Transcending to higher dimensions.
- Isogeny group actions.
- Signatures from isogenies.

Elliptic curves (picture over $\mathbb{R}$)



The elliptic curve $y^2 = x^3 - x + 1$ over $\mathbb{R}$.

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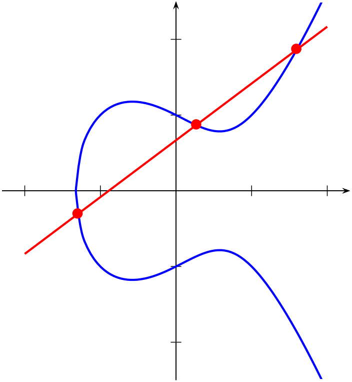

Addition law:

$$P + Q + R = \infty \quad \Longleftrightarrow \quad \{P, Q, R\} \text{ on a straight line.}$$

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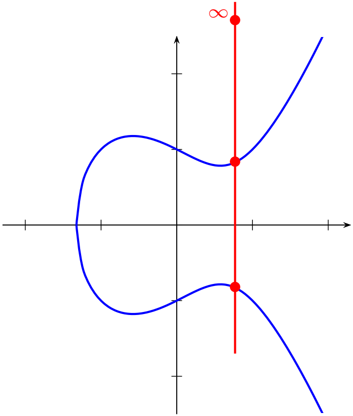
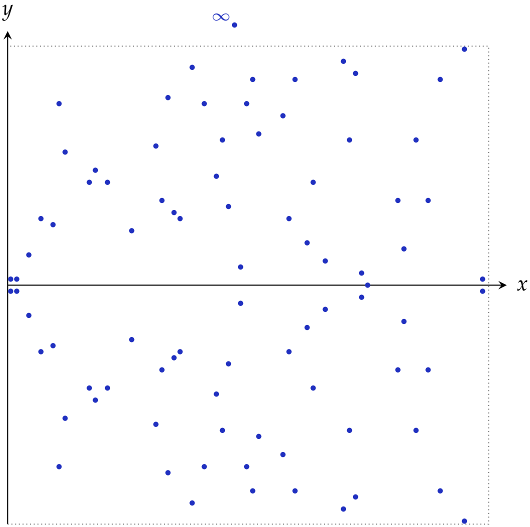


The *point at infinity* ∞ lies on every vertical line.

Elliptic curves (picture over $\mathbb{F}_p$)



The same curve $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

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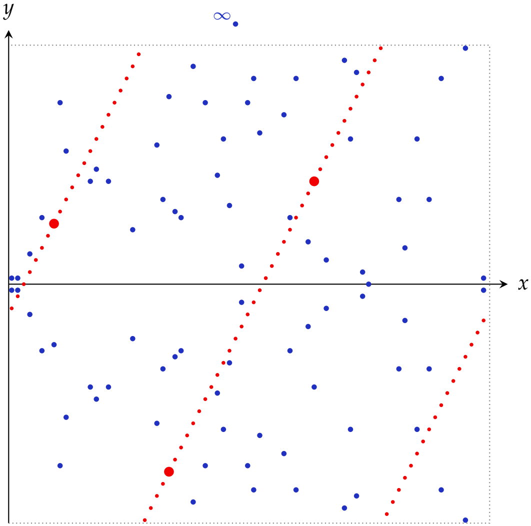

The addition law of $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

Isogenies

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...are just fancily-named

nice maps

between elliptic curves.

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The degree of a separable* isogeny is the size of its kernel.

Generic example: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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$\rightsquigarrow$ Decompose large-degree isogenies into prime steps.
That is, walk in an isogeny graph.

Computing isogenies: Vélu's formulas (1971)

Let G be a finite subgroup of an elliptic curve E. Then

$$P \mapsto \Bigg(x(P) + \sum_{Q \in G \setminus \{\infty\}} \big(x(P+Q) - x(Q) \big), \\ y(P) + \sum_{Q \in G \setminus \{\infty\}} \big(y(P+Q) - y(Q) \big) \Bigg)$$

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- computing the defining equation of E/G;
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This leads to formulas for

- computing the defining equation of E/G;
- evaluating the isogeny $E \to E/G$ at a point.

Complexity: $\Theta(\#G)$ $\rightsquigarrow$ only suitable for small degrees.

The $\sqrt{\text{é}}$lu algorithm reduces the cost to $\widetilde{\mathcal{O}}(\sqrt{\#G})$.

“Computing an isogeny”

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Keep in mind: Constructing isogenies $E \to \_$ is (usually) easy, constructing an isogeny $E \to E'$ given (E, E') is (usually) hard.

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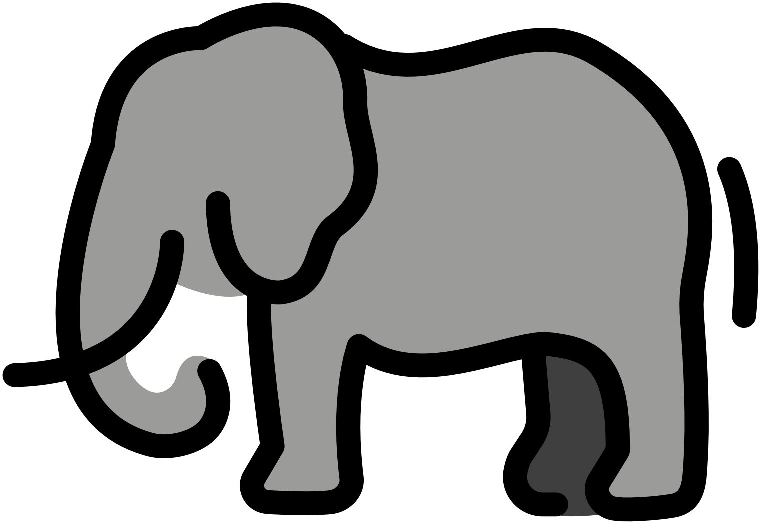
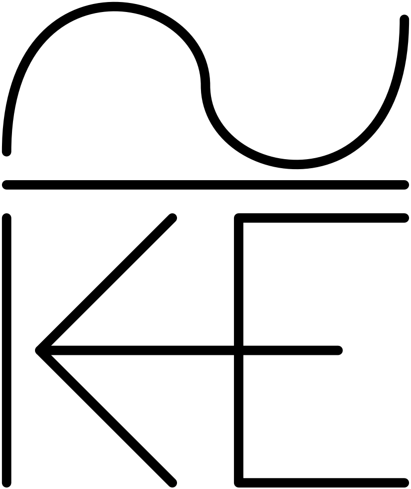


SIDH/SIKE



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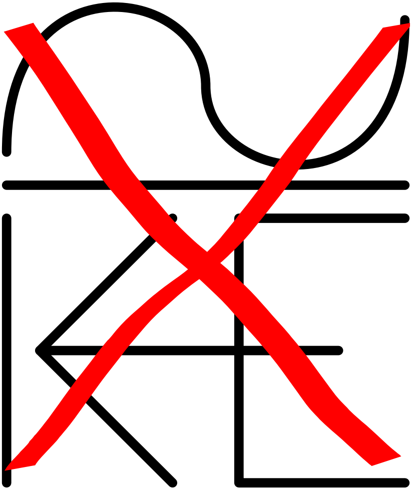

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It was catastrophically broken in 2022.

Isogeny-based key exchange: High-level view

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$$\begin{tikzcd}
E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E_A \\
E_B &
\end{tikzcd}$$

- Alice & Bob pick secret $\varphi_A : E \to E_A$ and $\varphi_B : E \to E_B$.
  (These isogenies correspond to walking on the isogeny graph.)

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$$\begin{tikzcd}
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- Alice and Bob transmit the end curves E_A and E_B.

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How to find "parallel" isogenies?

$$\begin{tikzcd}
E \arrow[rr, "\varphi_A"] \arrow[dd, swap, "\varphi_B"] & & E_A \arrow[d, thick, "\varphi'_B"] \\
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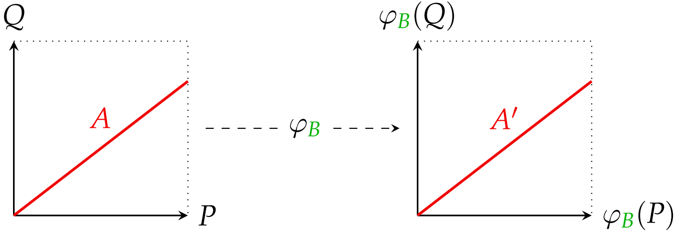

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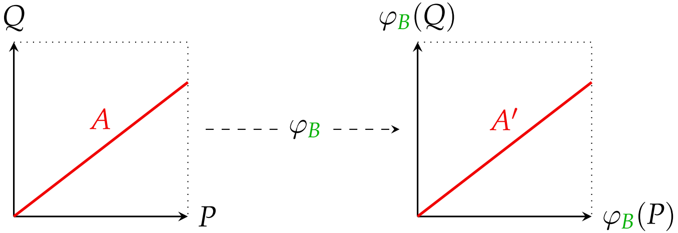

- Alice picks A as $\langle P + [a]Q \rangle$ for fixed public $P, Q \in E$.
- Bob includes $\varphi_B(P)$ and $\varphi_B(Q)$ in his public key.

$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$.

(Similarly for Bob.)

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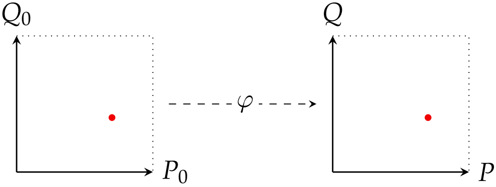

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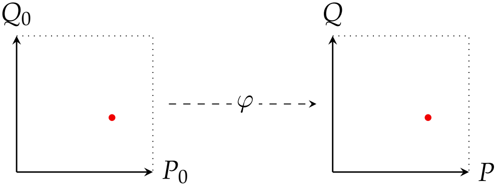

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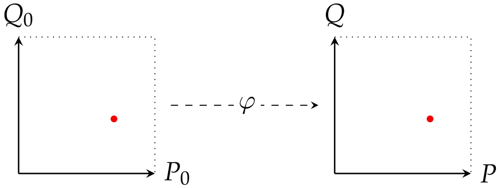

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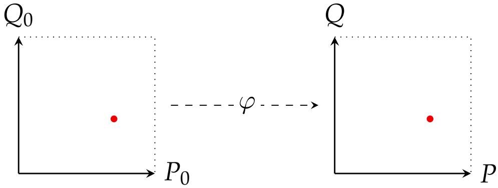

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⇝ The best thing to ever happen to isogenies! 🎉

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Main technique underlying attack:

Computing isogenies between *products* of elliptic curves

- The product $E \times E'$ is an abelian *surface*.
- Similar to elliptic curves in many ways:
  - Points form an abelian group.
  - Similar group structure, but more components.
  - Can define isogenies from kernel subgroups.

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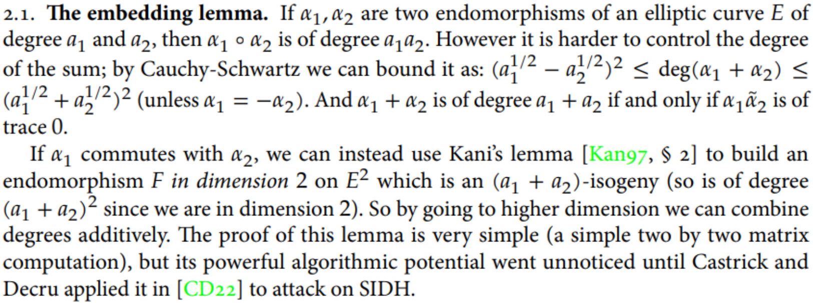

2.1. **The embedding lemma.** If α_1, α_2 are two endomorphisms of an elliptic curve E of degree a_1 and a_2, then $\alpha_1 \circ \alpha_2$ is of degree $a_1 a_2$. However it is harder to control the degree of the sum; by Cauchy-Schwartz we can bound it as: $(a_1^{1/2} - a_2^{1/2})^2 \leq \deg(\alpha_1 + \alpha_2) \leq (a_1^{1/2} + a_2^{1/2})^2$ (unless $\alpha_1 = -\alpha_2$). And $\alpha_1 + \alpha_2$ is of degree $a_1 + a_2$ if and only if $\alpha_1 \tilde{\alpha}_2$ is of trace 0.

If α_1 commutes with α_2, we can instead use Kani's lemma [Kan97, § 2] to build an endomorphism F *in dimension* 2 on E^2 which is an $(a_1 + a_2)$-isogeny (so is of degree $(a_1 + a_2)^2$ since we are in dimension 2). So by going to higher dimension we can combine degrees additively. The proof of this lemma is very simple (a simple two by two matrix computation), but its powerful algorithmic potential went unnoticed until Castrick and Decru applied it in [CD22] to attack on SIDH.

— Damien Robert [ePrint 2022/1704]

The embedding lemma

Consider a commutative diagram of isogenies

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\psi"'] & E' \arrow[d, "\psi'"] \\ E'' \arrow[r, "\varphi'"'] & E''' \end{tikzcd}$$

where $a := \deg \varphi$ and $b := \deg \psi$ are coprime, and let $N := a + b$.

Lemma. Then

$$\Phi := \begin{pmatrix} \varphi & \widehat{\psi}' \\ -\psi & \widehat{\varphi}' \end{pmatrix} \colon (P, Q) \mapsto \big(\varphi(P) + \widehat{\psi}'(Q), -\psi(P) + \widehat{\varphi}'(Q)\big)$$

defines an N-isogeny $E \times E''' \to E' \times E''$.

Its kernel is $\ker(\Phi) = \big\{(\widehat{\varphi}(T), \psi'(T)) \mid T \in E'[N]\big\}$.

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...is an efficient representation of *any (!)* isogeny between two elliptic curves.

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⚠ Requires isogeny formulas for principally polarized abelian varieties of dimension ≥ 2. Highly non-trivial matter, but fundamentally doable and efficient.

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CSIDH’s solution:

Use special isogenies φ_A which can be transported to the curve E_B totally independently of the secret isogeny φ_B.

(Similarly with reversed roles, of course.)

CSIDH [ˈsiːˌsaɪd]

[Castryck–Lange–Martindale–Panny–Renes 2018]

“Special” isogenies

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$\leadsto$ For all such E can canonically find an isogeny $\varphi_\ell \colon E \to E'$.

We consider prime ℓ and refer to φ_ℓ as a “special” isogeny.

Cycles from "special" isogenies

What happens when we iterate such a "special" isogeny?

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$$\begin{tikzcd}
 & & & & & & \cdots \arrow[dll] & E_{\ell^6} \arrow[l] \\
 & & & & E_{\ell^{r-1}} \arrow[dl] & & & \\
 & & & E_{\ell^r} \arrow[dd] & & & & E_{\ell^5} \arrow[uu] \\
 & & & & & E_{\ell^4} \arrow[urr] & & \\
E \arrow[r] & E_\ell \arrow[r] & E_{\ell^2} \arrow[r] & E_{\ell^3} \arrow[urr] & & & &
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 & & & E_{\ell^r} \arrow[dd] & & & E_{\ell^5} \arrow[u] \\
 & & & & & E_{\ell^4} \arrow[ur] & \\
E \arrow[r] & E_{\ell} \arrow[r] & E_{\ell^2} \arrow[r] & E_{\ell^3} \arrow[urr] & & &
\end{tikzcd}$$

- Fact: Each curve has only one other rational ℓ-isogeny.

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What happens when we iterate such a “special” isogeny?

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 & & & & & \cdots \arrow[dll] & E_{\ell^6} \arrow[l] \\
 & & & E_{\ell^{r-1}} \arrow[d] & & & \\
 & & & E_{\ell^r} \arrow[dd] & & & E_{\ell^5} \arrow[uu] \\
 & & & & E_{\ell^4} \arrow[urr] & & \\
E \arrow[r] & E_{\ell} \arrow[r] & E_{\ell^2} \arrow[r] & E_{\ell^3} \arrow[ur] & & &
\end{tikzcd}$$

- Fact: Each curve has only one other rational ℓ-isogeny.

!! Reverse arrows are unique; the “tail” $E \to E_{\ell^3}$ cannot exist.

Cycles from “special” isogenies

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$\Longrightarrow$ The “special” isogenies φ_ℓ form isogeny cycles!

Compatible cycles from “special” isogenies

What happens when we compose those “special” isogenies?

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$$\begin{tikzcd}
 & & E_{m\ell} & E_{\ell m} \\
E_m \arrow[urr, "\varphi'_\ell"] & & & \\
 & & E_\ell \arrow[uur, "\varphi_m"] & \\
E \arrow[uu, "\varphi'_m"] \arrow[uurr, "\varphi_\ell"] & & &
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- Fact: $\ker(\varphi'_\ell \circ \varphi'_m) = \ker(\varphi_m \circ \varphi_\ell) = \langle \ker \varphi_\ell, \ker \varphi'_m \rangle$.

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!! The order cannot matter $\Longrightarrow$ cycles must be compatible.

CSIDH in one slide

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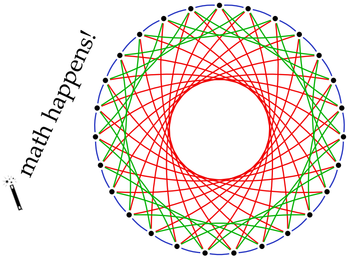

$p = 419$
$\ell_1 = 3$
$\ell_2 = 5$
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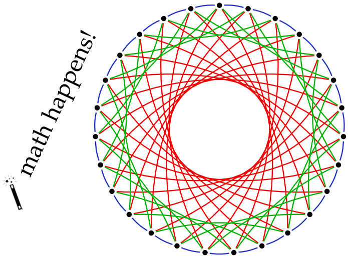

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$\ell_3 = 7$

- Walking "left" and "right" on any ℓ_i-subgraph is efficient.

Walking in the CSIDH graph (in SageMath)

```
sage: E = EllipticCurve(GF(419^2), [1,0])
sage: E
Elliptic Curve defined by y^2 = x^3 + x
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....:     try:
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sage: P.order().factor()
2 * 3 * 7
sage: EE = E.isogeny_codomain(2*3*P)  # "left" 7-step
sage: EE
Elliptic Curve defined by y^2 = x^3 + 285*x + 87
        over Finite Field in z2 of size 419^2
```
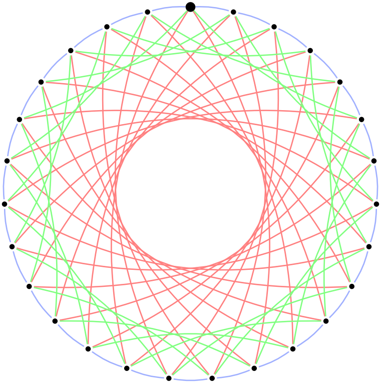

CSIDH key exchange

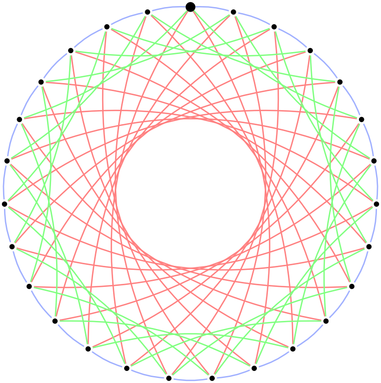
Alice

[+, +, −, −]



Bob

[−, +, −, −]



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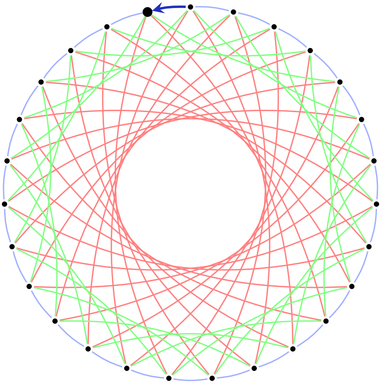

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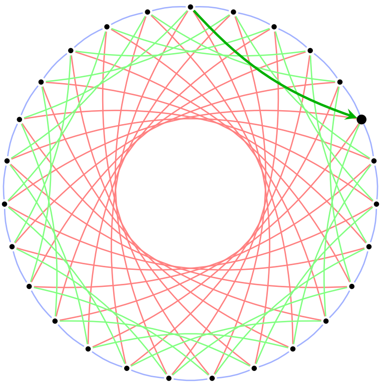

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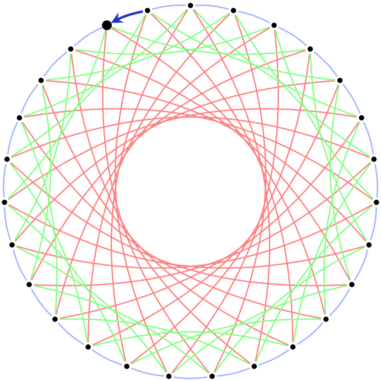

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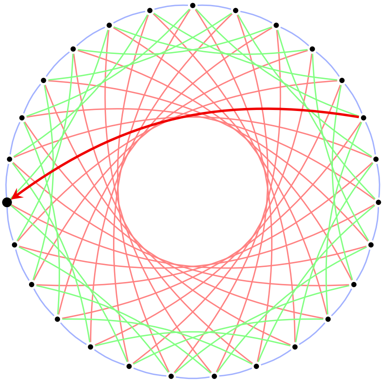

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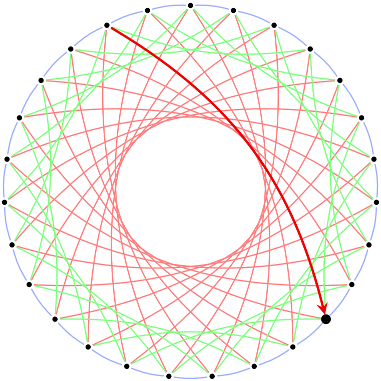

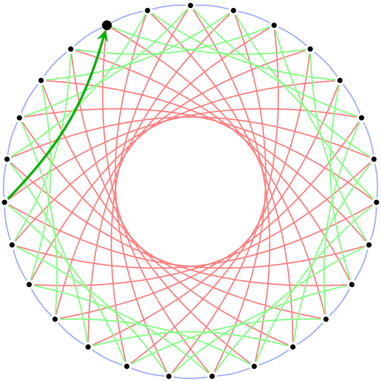

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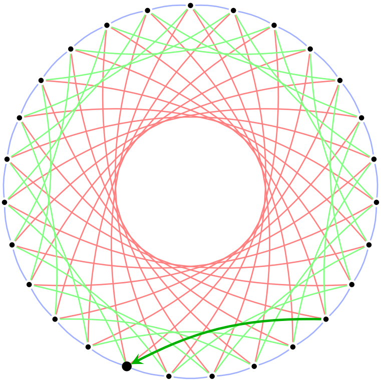

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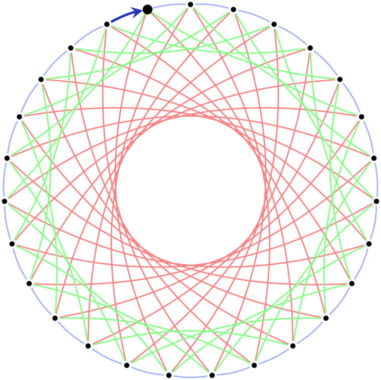

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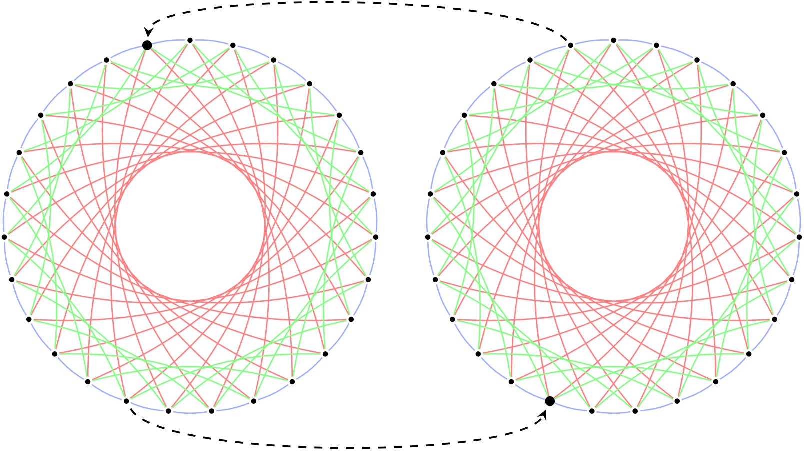

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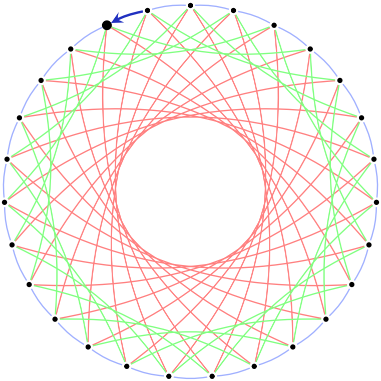

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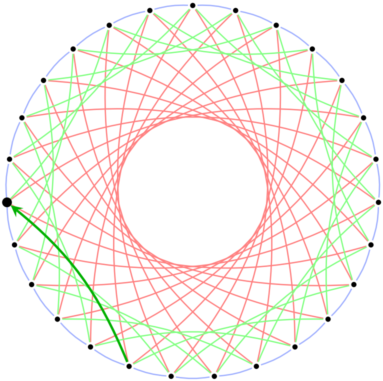

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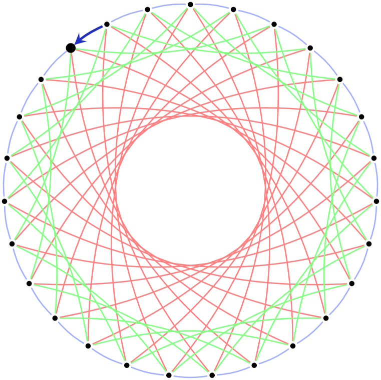

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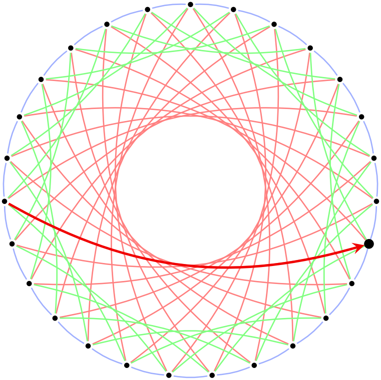

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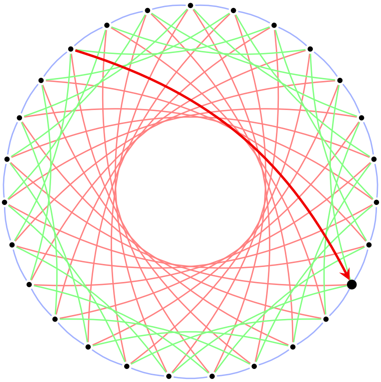

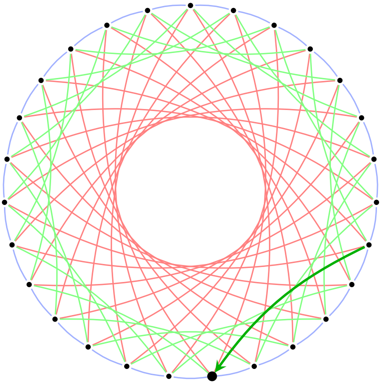

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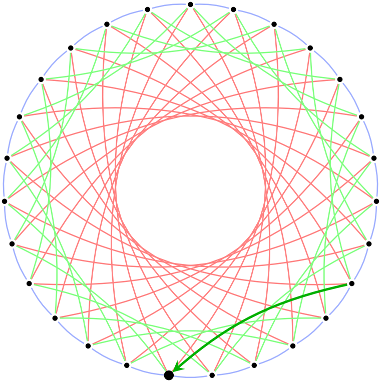

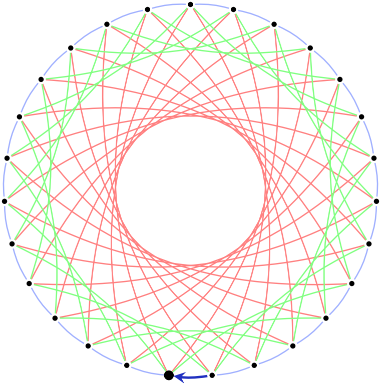

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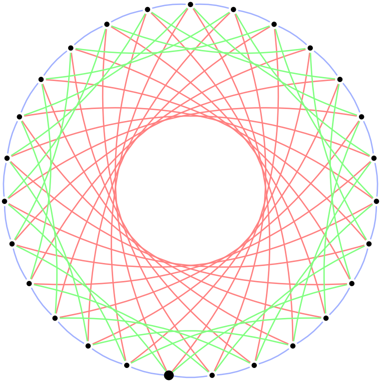

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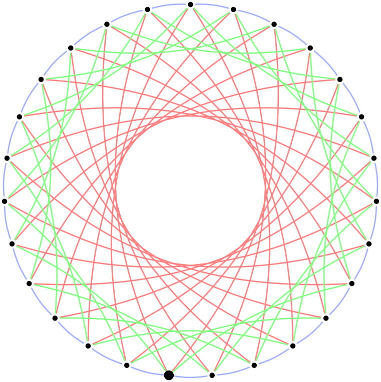

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The lattice Λ is computable in subexponential time classically, and in polynomial time using a quantum computer.
It is used to construct more advanced schemes (*"CSI-FiSh"*).

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- 2023: "Clapoti" — a polynomial-time algorithm for arbitrary combinations of operations in the group and evaluations of the action. $\rightsquigarrow$ *"KLaPoTi"*, *"PEGASIS"*.
(Previously, only restricted sequences of operations were efficient.)

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$\Longrightarrow$ Security estimates for CSIDH & friends vary wildly. ⚠

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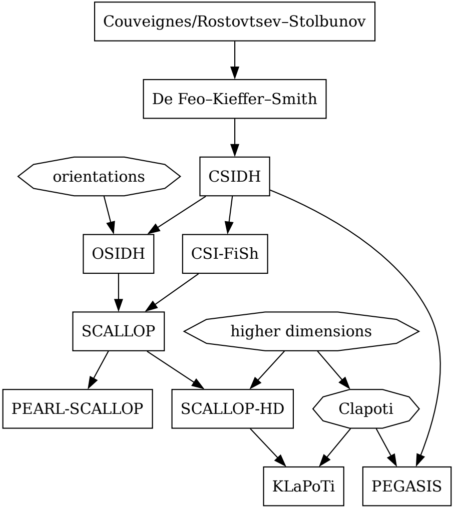

Plan for this talk

- Elliptic curves & isogenies. ✓
- The SIKE attacks. ✓
- Transcending to higher dimensions. ✓
- Isogeny group actions. ✓
- Signatures from isogenies.

SQIsign: What?

https://sqisign.org

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- A new-ish and very hot post-quantum signature scheme.
- Based on super cool mathematics. ☺

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☺ We understand the structure of $\mathrm{End}(E)$.

More “special” isogenies

- Earlier: “Special” isogenies φ_ℓ with rational kernel points.
- In other words: $\ker \varphi_\ell = \ker[\ell] \cap \ker(\pi - 1)$.
  (Here π is the Frobenius endomorphism $\pi\colon (x, y) \mapsto (x^p, y^p)$.)

!! Over $\mathbb{F}_{p^2}$, we can have more endomorphisms.
Example: $y^2 = x^3 + x$ has $\iota\colon (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$.

- Extremely non-obvious fact in this setting:

Every isogeny $\varphi\colon E \to E'$ comes from an ideal $I_\varphi \subseteq \mathrm{End}(E)$.

☺ We understand the structure of $\mathrm{End}(E)$.

☺ We understand how I_φ, I_ψ relate for isogenies $\varphi, \psi\colon E \to E'$.

$\rightsquigarrow$ one-sided ideal class *set* of $\mathrm{End}(E)$, etc.

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Theorem. Fix E_0 supersingular. The (contravariant) functor

$$E \longmapsto \mathrm{Hom}(E, E_0)$$

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☺ One direction is easy, the other seems hard! $\rightsquigarrow$ *Cryptography!*

The Deuring correspondence (examples)

Let $p = 7799999$ and let $\mathbf{i}, \mathbf{j}$ satisfy $\mathbf{i}^2 = -1,\ \mathbf{j}^2 = -p,\ \mathbf{ji} = -\mathbf{ij}$.

The ring $\mathcal{O}_0 = \mathbb{Z} \oplus \mathbb{Z}\,\mathbf{i} \oplus \mathbb{Z}\,\frac{\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{1+\mathbf{ij}}{2}$
corresponds to the curve $E_0\colon\ y^2 = x^3 + x$.

The ring $\mathcal{O}_1 = \mathbb{Z} \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+32631010\mathbf{i}+\mathbf{ij}}{9894}$
corresponds to the curve $E_1\colon\ y^2 = x^3 + 1$.

The ideal $I = \mathbb{Z}\,4947 \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{598+4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+598\mathbf{i}+\mathbf{ij}}{2}$
defines an isogeny $E_0 \to E_1$ of degree $4947 = 3 \cdot 17 \cdot 97$.

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Constructively, *partially* known endomorphism rings are useful.
$\rightsquigarrow$ **Oriented curves** and **isogeny group actions**. 🏝

Signing with isogenies à la SQIsign

- Fiat–Shamir: signature scheme from identification scheme.

$$\begin{tikzcd} E_0 \arrow[r, dashed, "secret"] & E_{pk} \end{tikzcd}$$

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- It relies on an explicit form of the Deuring correspondence.

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Via the Deuring correspondence:

- From $\mathrm{End}(E), \mathrm{End}(E')$, can randomize within $\mathrm{Hom}(E, E')$.

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> *“If you have KLPT implemented very nicely as a black box, then anyone can implement SQIsign.”* — Yan Bo Ti

SQIsign: Why?

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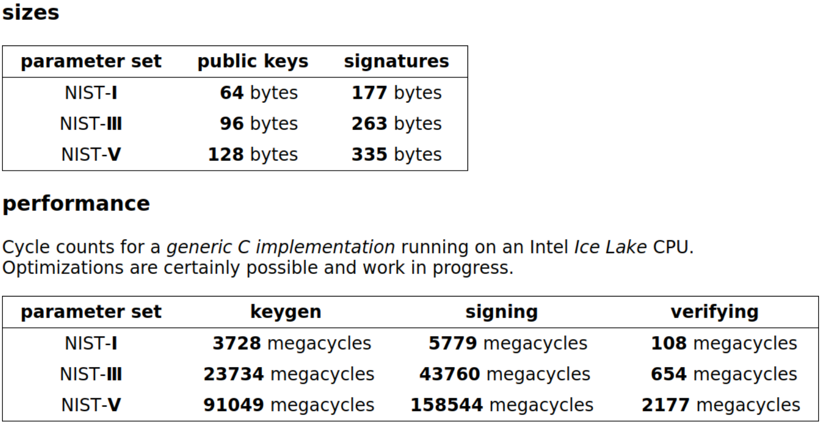

sizes

parameter set	public keys	signatures
NIST-**I**	**64** bytes	**177** bytes
NIST-**III**	**96** bytes	**263** bytes
NIST-**V**	**128** bytes	**335** bytes

performance

Cycle counts for a *generic C implementation* running on an Intel *Ice Lake* CPU.
Optimizations are certainly possible and work in progress.

parameter set	keygen	signing	verifying
NIST-**I**	**3728** megacycles	**5779** megacycles	**108** megacycles
NIST-**III**	**23734** megacycles	**43760** megacycles	**654** megacycles
NIST-**V**	**91049** megacycles	**158544** megacycles	**2177** megacycles

Source: `https://sqisign.org` (2023–2024)

SQIsign (current version): Dramatically improved!

- The $\geq 20\times$ speedup over the original version of SQIsign comes from the new tools underlying the SIKE attacks.
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Main <u>idea</u> (from "SQIsign[H2]D" papers): Use HD representation.

$$\begin{tikzcd}
E_0 \arrow[r, Rightarrow, dashed, "secret"] \arrow[d, Rightarrow, dashed, "commitment"'] & E_{pk} \arrow[d, Rightarrow, "response"] \\
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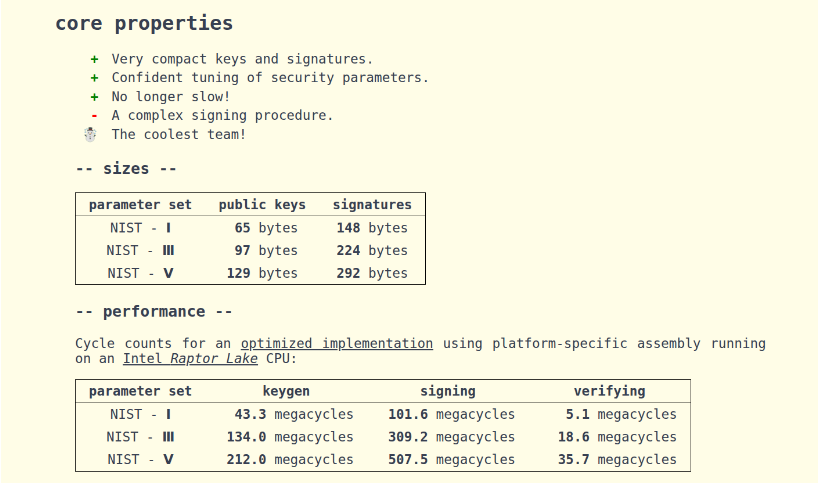
$\longrightarrow$ 1-dimensional isogeny $\Longrightarrow$ 2-dimensional isogeny

SQIsign (current version): Numbers



core properties

- + Very compact keys and signatures.
- + Confident tuning of security parameters.
- + No longer slow!
- - A complex signing procedure.
- ☃ The coolest team!

-- sizes --

parameter set	public keys	signatures
NIST - **I**	**65** bytes	**148** bytes
NIST - **III**	**97** bytes	**224** bytes
NIST - **V**	**129** bytes	**292** bytes

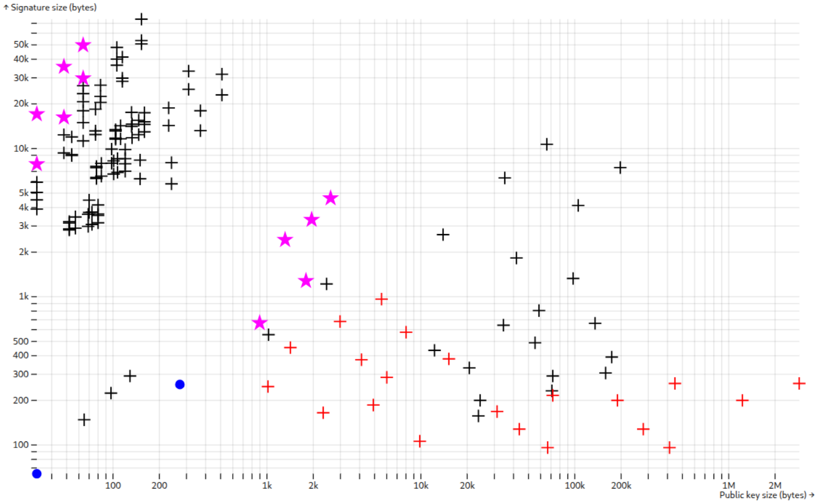
-- performance --

Cycle counts for an optimized implementation using platform-specific assembly running on an Intel *Raptor Lake* CPU:

parameter set	keygen	signing	verifying
NIST - **I**	**43.3** megacycles	**101.6** megacycles	**5.1** megacycles
NIST - **III**	**134.0** megacycles	**309.2** megacycles	**18.6** megacycles
NIST - **V**	**212.0** megacycles	**507.5** megacycles	**35.7** megacycles

Source: `https://sqisign.org` (2025–?)

SQIsign (current version): Comparison



Source: `https://pqshield.github.io/nist-sigs-zoo`

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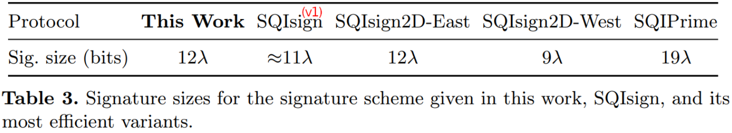
- **Challenge:** A large prime q.
- **Response:** An isogeny $\varphi\colon E \to \_$ of degree q.
  How? Create HD representation of φ using knowledge of $\mathrm{End}(E)$!

PRISM: Parameters



Protocol	**This Work**	SQIsign$^{(v1)}$	SQIsign2D-East	SQIsign2D-West	SQIPrime
Sig. size (bits)	12λ	$\approx 11\lambda$	12λ	9λ	19λ

Table 3. Signature sizes for the signature scheme given in this work, SQIsign, and its most efficient variants.

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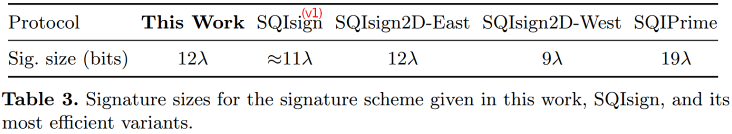

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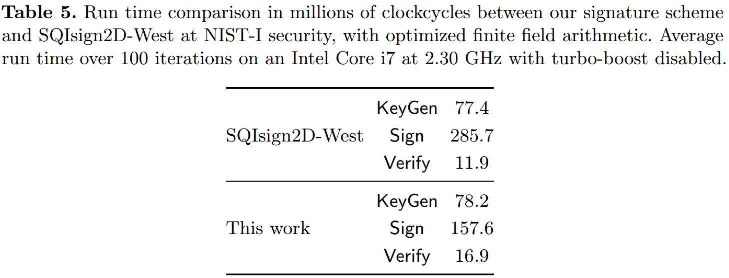

Table 5. Run time comparison in millions of clockcycles between our signature scheme and SQIsign2D-West at NIST-I security, with optimized finite field arithmetic. Average run time over 100 iterations on an Intel Core i7 at 2.30 GHz with turbo-boost disabled.

SQIsign2D-West	KeyGen	77.4
	Sign	285.7
	Verify	11.9
This work	KeyGen	78.2
	Sign	157.6
	Verify	16.9

Plan for this talk

- Elliptic curves & isogenies. ✓
- The SIKE attacks. ✓
- Transcending to higher dimensions. ✓
- Isogeny group actions. ✓
- Signatures from isogenies. ✓

Ad break

THE isogeny club

Seminar Sessions

A seminar session for young isogenists.

https://isogeny.club

Questions?

(Also feel free to email me: `lorenz@yx7.cc`)