Introduction to Isogeny-based Cryptography

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Why Isogeny-Based Crypto?

The discrete logarithm problem (DLP) is a fundamental building block in crypto:

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Let G be a group. For $g \in G$ and $n \in \mathbb{Z}$, given g and g^n, find n.

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One solution: Isogeny-based cryptography.

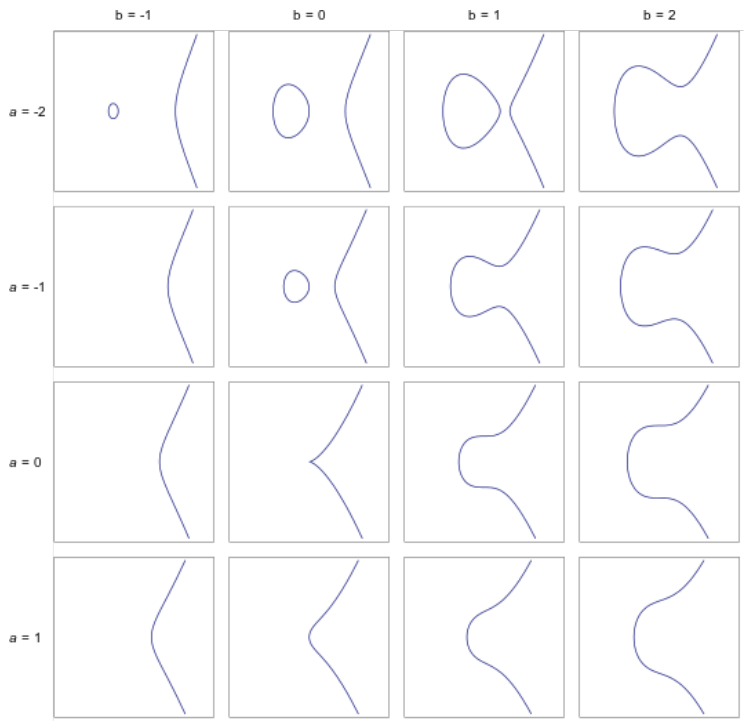
Fundamentals: elliptic curves

Definition

Let k be a field of characteristic $\neq 2$. An elliptic curve over k is a smooth[1] curve

$$E/k : y^2 = f(x),$$

where $f(x) \in k[x]$ is of degree 3.



[1] No self-intersections or cusps.

Fundamentals: the group law on elliptic curves

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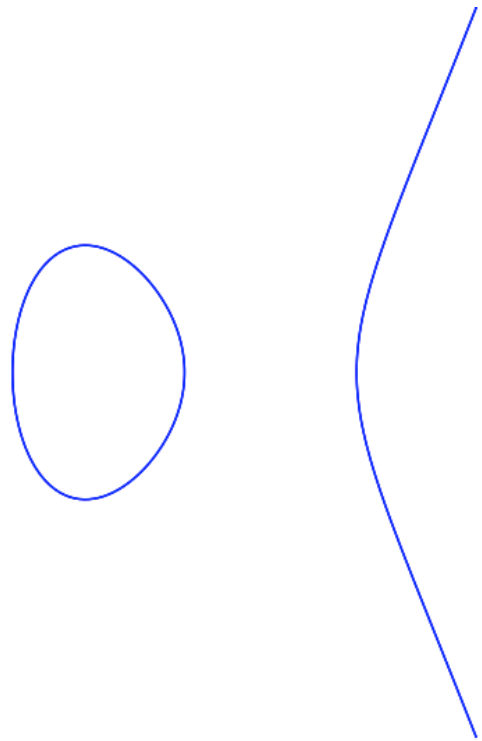

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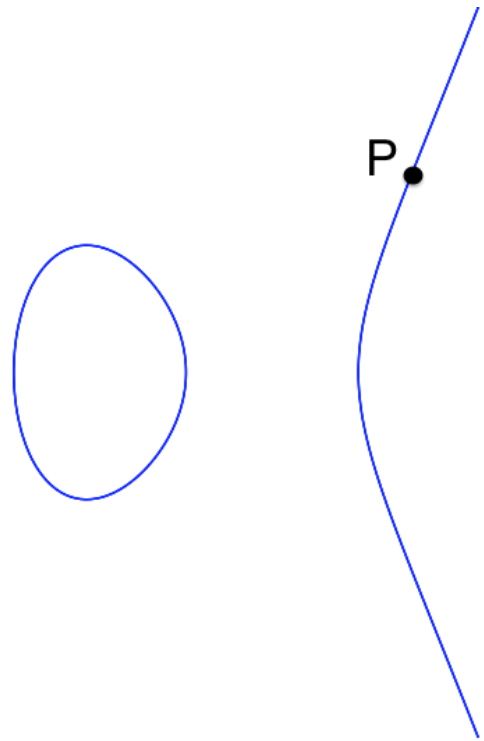

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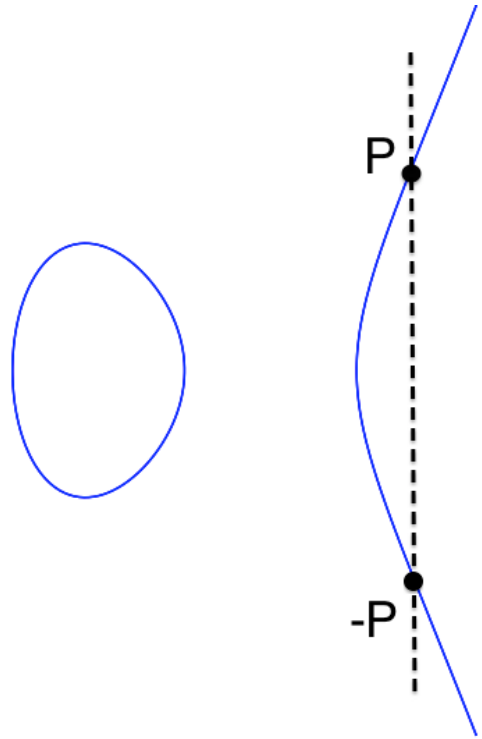

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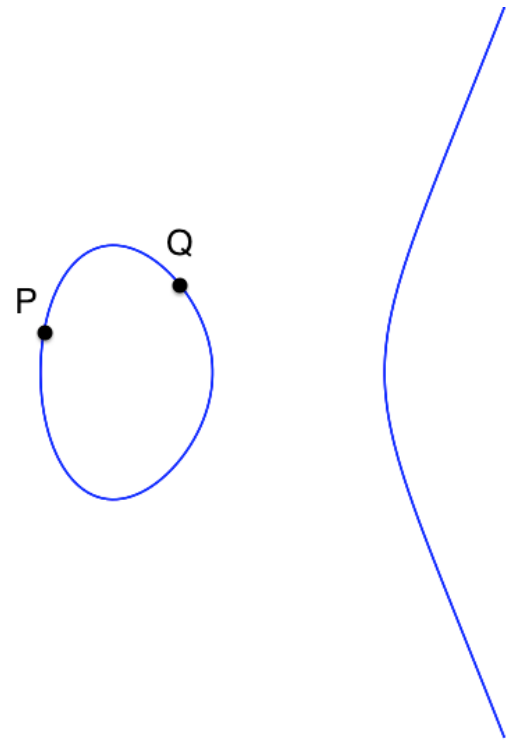

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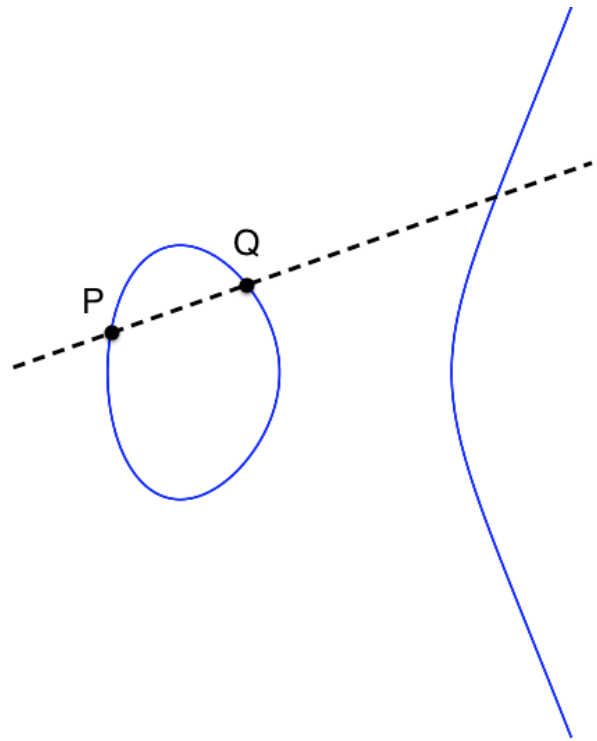

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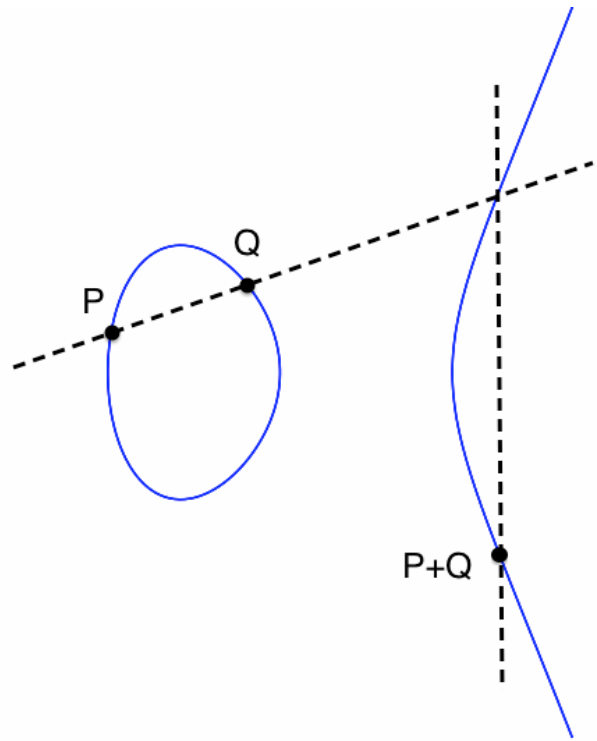

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Especially important for isogeny-based crypto:

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Let $E/\mathbb{F}_q$ be an elliptic curve, with $q = p^n$.
E is supersingular if $p \mid (q + 1 - \#E(\mathbb{F}_q))$.
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Important special cases:

- When $E/\mathbb{F}_p$ supersingular and $\#E(\mathbb{F}_p) = p + 1$.
- When $E/\mathbb{F}_{p^2}$ supersingular and $\#E(\mathbb{F}_{p^2}) = (p + 1)^2$.

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Example

Define $E/\mathbb{F}_5 : y^2 = x^3 + 1$. Then

$$E(\mathbb{F}_5) = \{(0, 1), (0, -1), (2, 3), (2, -3), (-1, 0), P_\infty\},$$

so $E/\mathbb{F}_5$ is supersingular.

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Define $E_{51}/\mathbb{F}_{419} : y^2 = x^3 + 51x^2 + x$

$$\begin{array}{rccc} [2]: & E_{51} & \to & E_{51} \\ & (x,y) & \mapsto & 2\cdot(x,y) := (x,y)+(x,y) \end{array}$$

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- Explicit calculations show that:

$$\begin{array}{rccc} [2]: & E_{51} & \to & E_{51} \\ & (x,y) & \mapsto & \left(\frac{\frac{1}{2}x^4-18x^3-163x^2-18x+\frac{1}{2}}{8x(x^2+9x+1)}, \frac{y(x^6+18x^5+5x^4-5x^2-18x-1)}{(8x(x^2+9x+1))^2}\right). \end{array}$$

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- A less obvious isogeny:

$$\begin{array}{rccc} f: & E_{51} & \to & E_9 \\ & (x,y) & \mapsto & \left(\frac{x^3-183x^2+73x+30}{(x+118)^2}, y\frac{x^3-65x^2-104x+174}{(x+118)^3}\right). \end{array}$$

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- Vélu's formulas compute the ℓ-isogeny from its kernel in time $\Theta(\ell)$.

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$\rightsquigarrow$ we could also write

$$f = \varphi_{\langle(-118,51)\rangle} : E_{51} \to E_{51}/\langle(-118, 51)\rangle.$$

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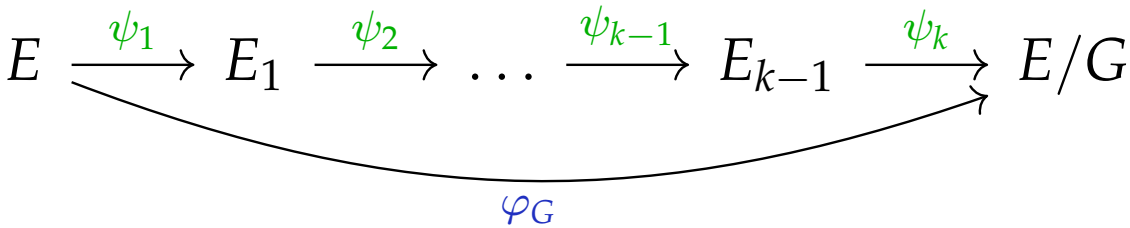

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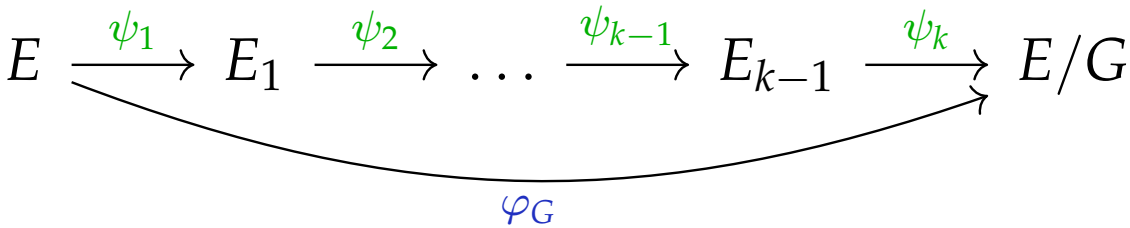

$\rightsquigarrow$ Complexity: $O(k^2 \cdot \ell)$. *Exponentially smaller* than $\#G = \ell^k$!
'Optimal strategy' improves this to $O(k \log k \cdot \ell)$.

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(As before $[\ell]$ denotes the multiplication-by-ℓ map.)

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- Nodes: elliptic curves defined over $\mathbb{F}_q$ with n points (up to $\mathbb{F}_q$-isomorphism).
- Edges: an edge $E - E'$ represents an ℓ-isogeny $E \to E'$ defined over $\mathbb{F}_q$ together with its dual isogeny. (up to post-composition with isomorphisms).

Isogeny graphs

Example

Replace eg. 

by

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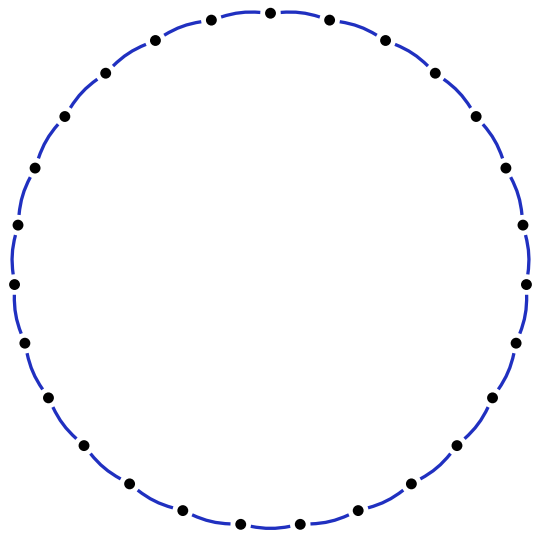
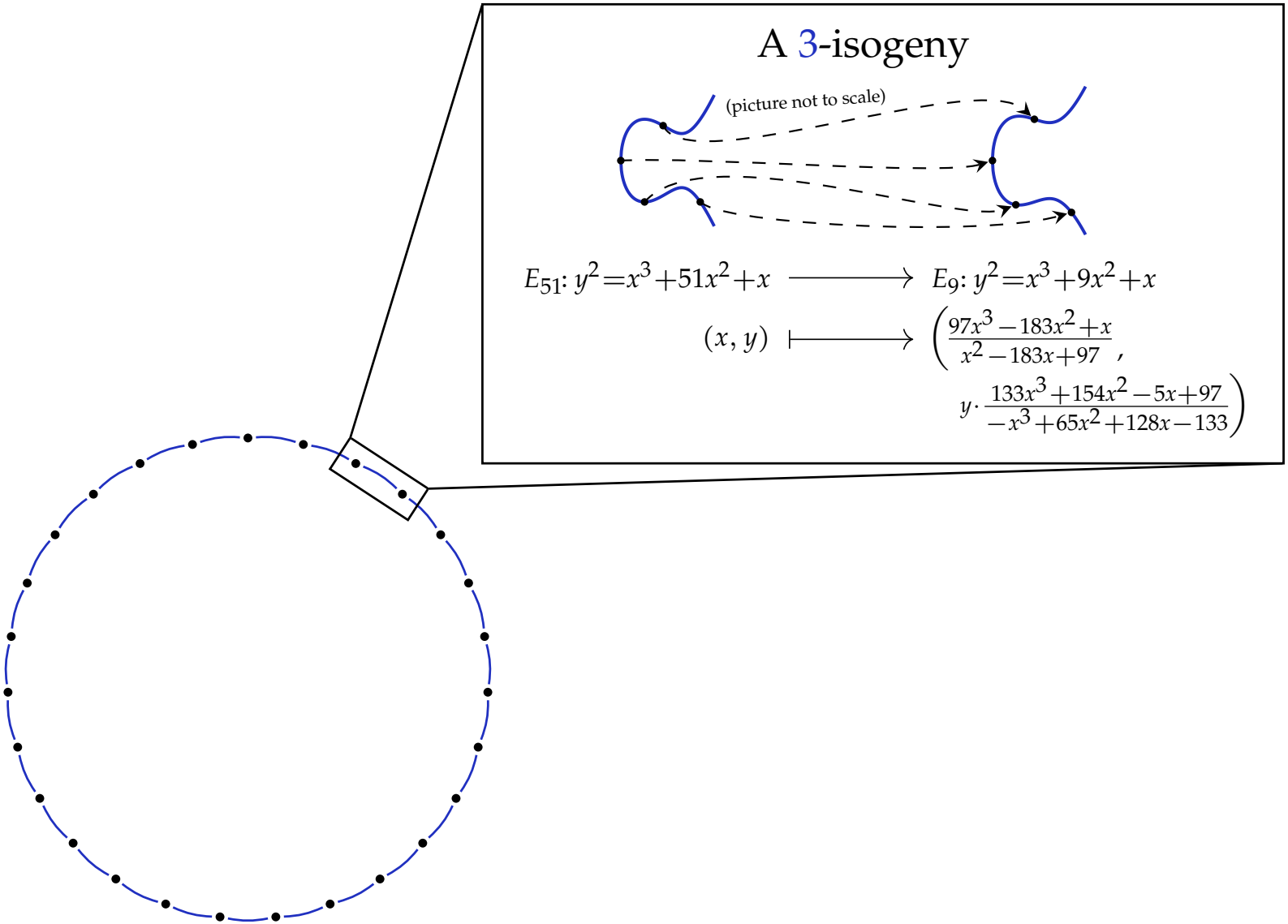
Then the graph $G_{3,420}$ over $\mathbb{F}_{419}$ looks like:



[NB: the nodes with $p + 1 = 420$ points are the supersingular nodes].

Isogeny graphs



Fragen?

Questions?

Domande?

Dumondas?

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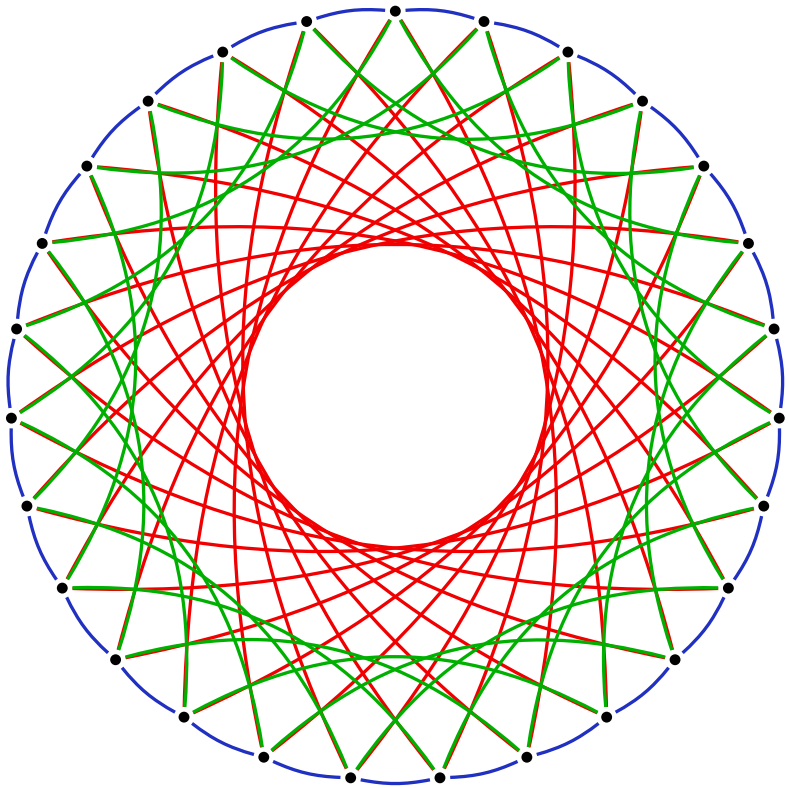
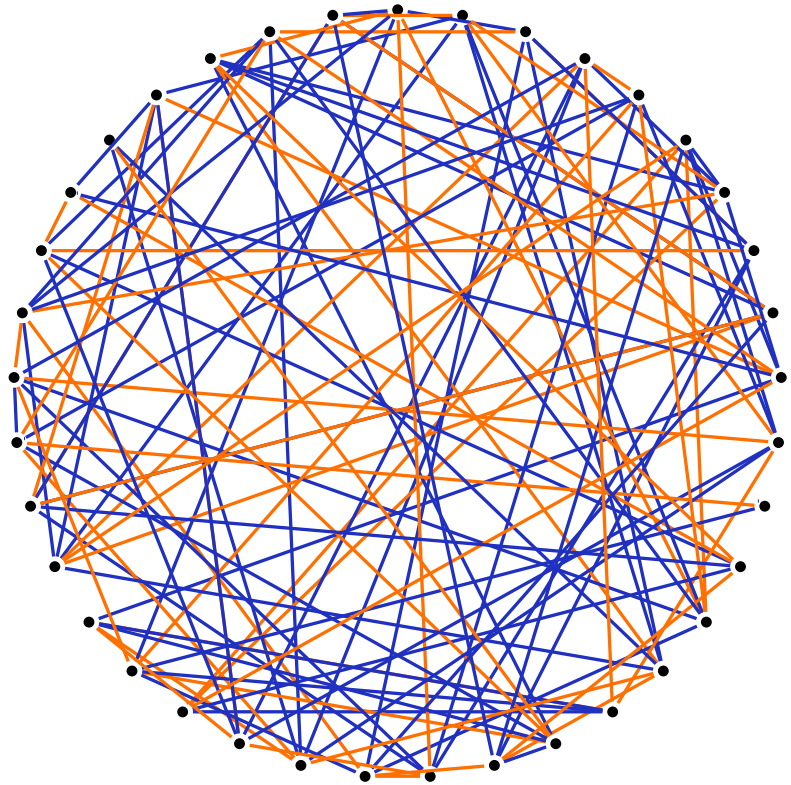
It is easy to construct graphs that satisfy *almost* all of these —
not enough for crypto!

The beauty and the beast

Components of well-chosen isogeny graphs look like this:





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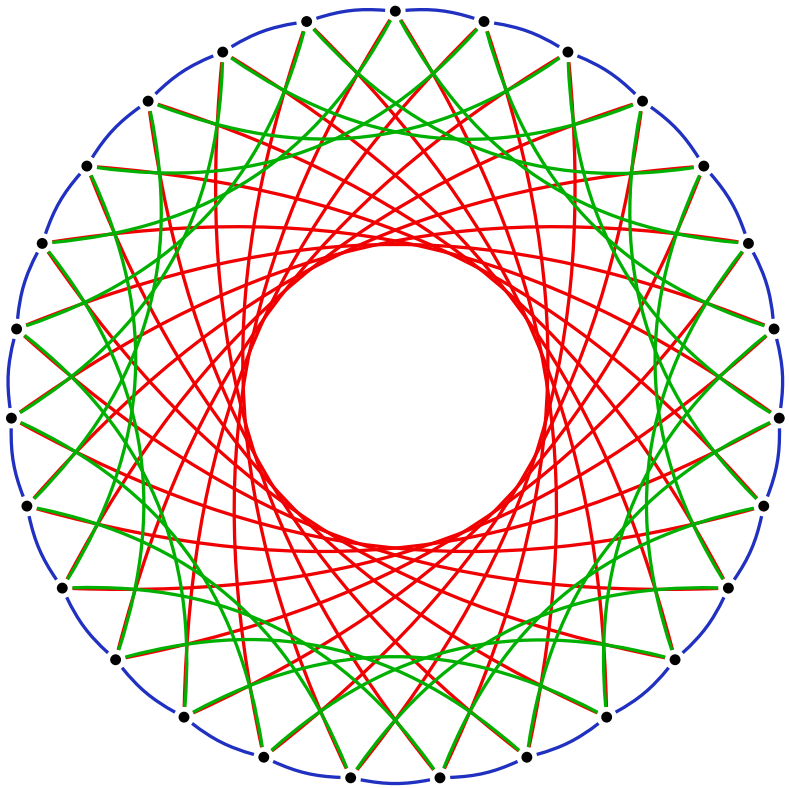

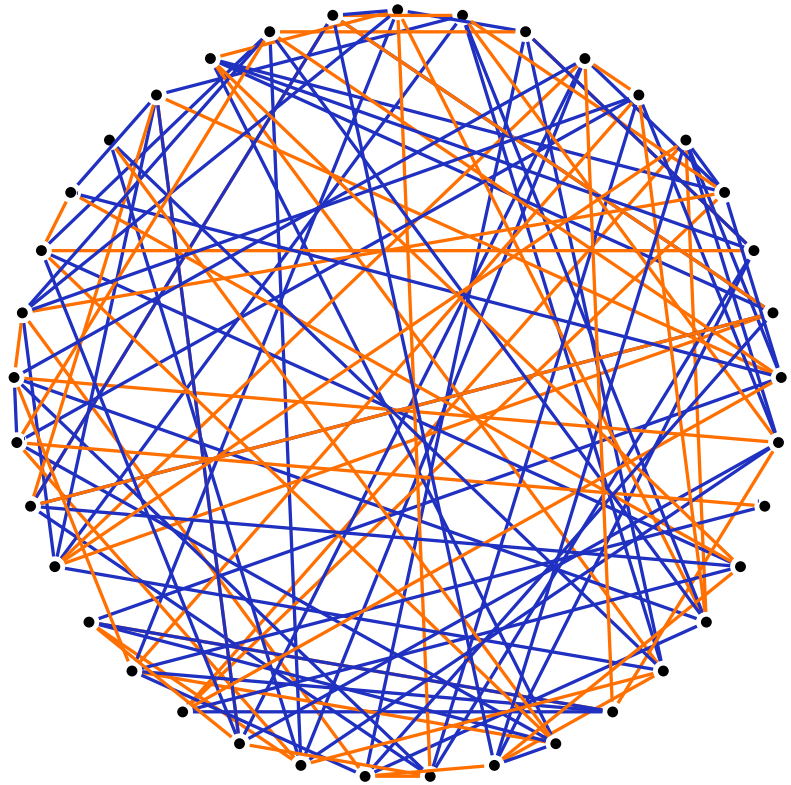

Which of these is good for crypto?

The beauty and the beast

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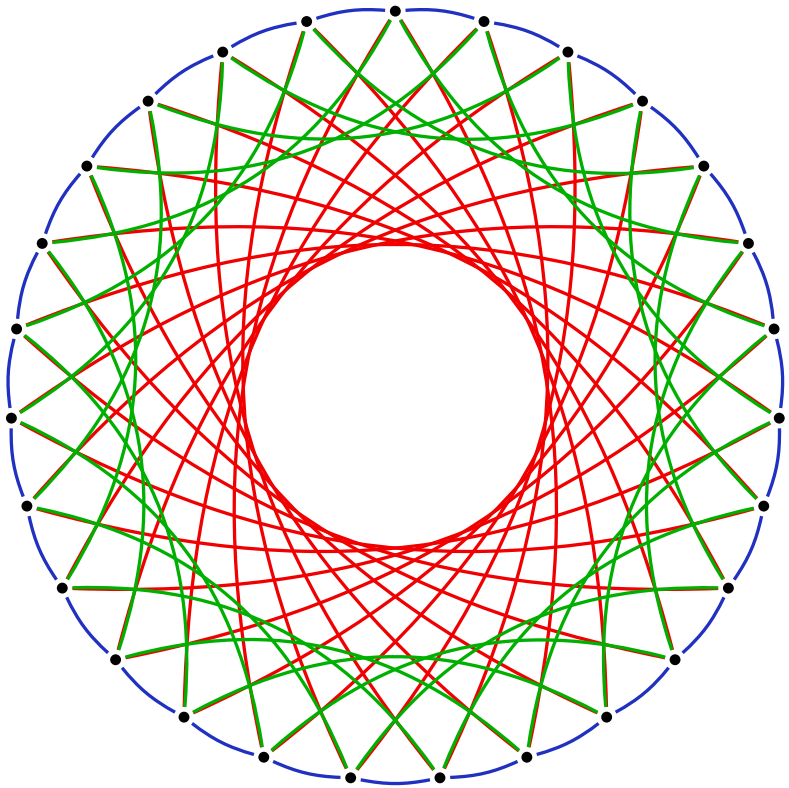
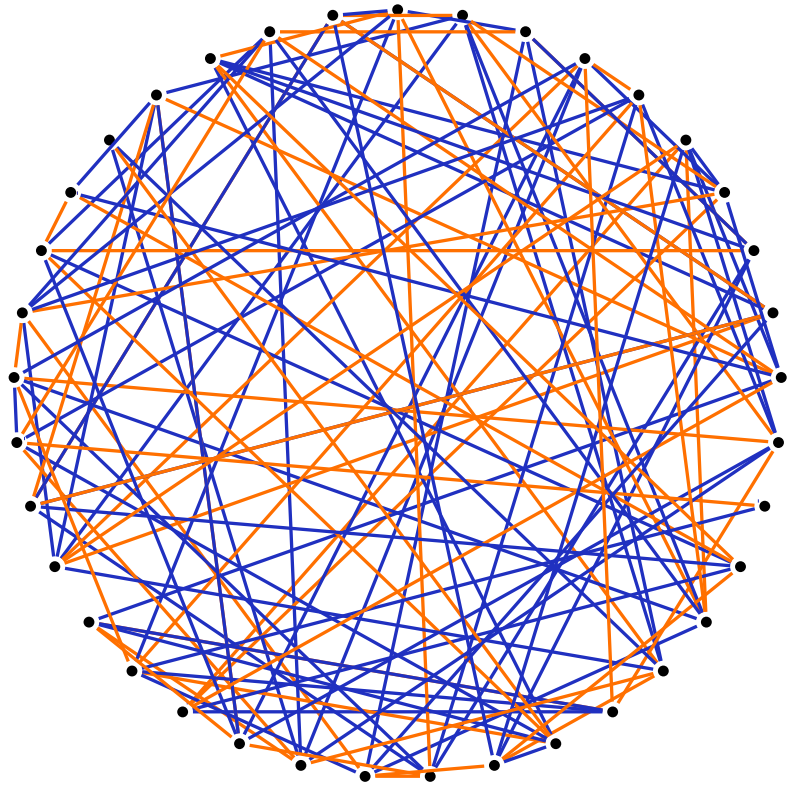
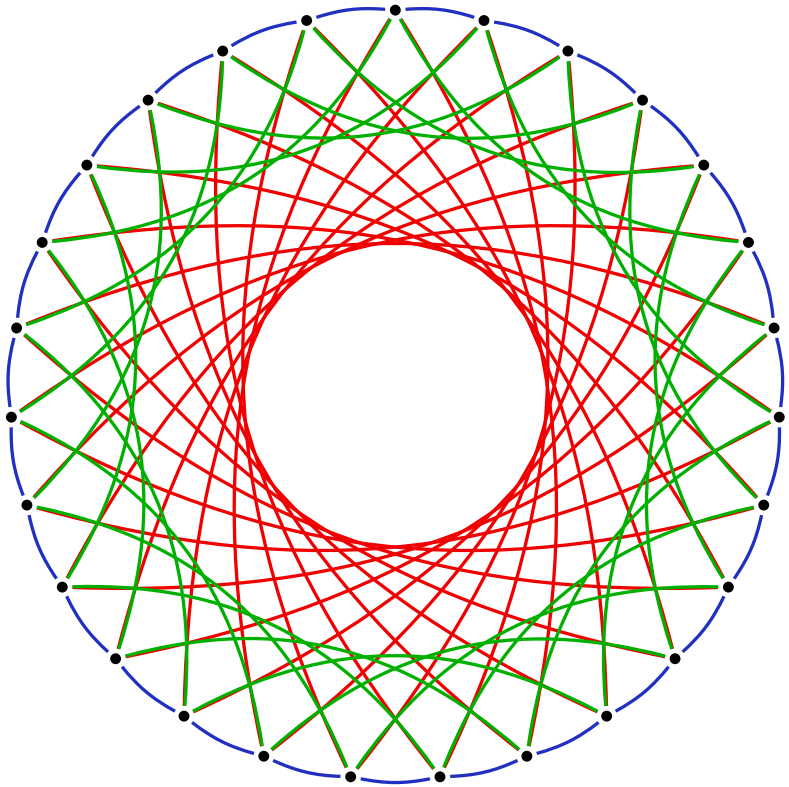




Which of these is good for crypto? **Both.**

The beauty and the beast

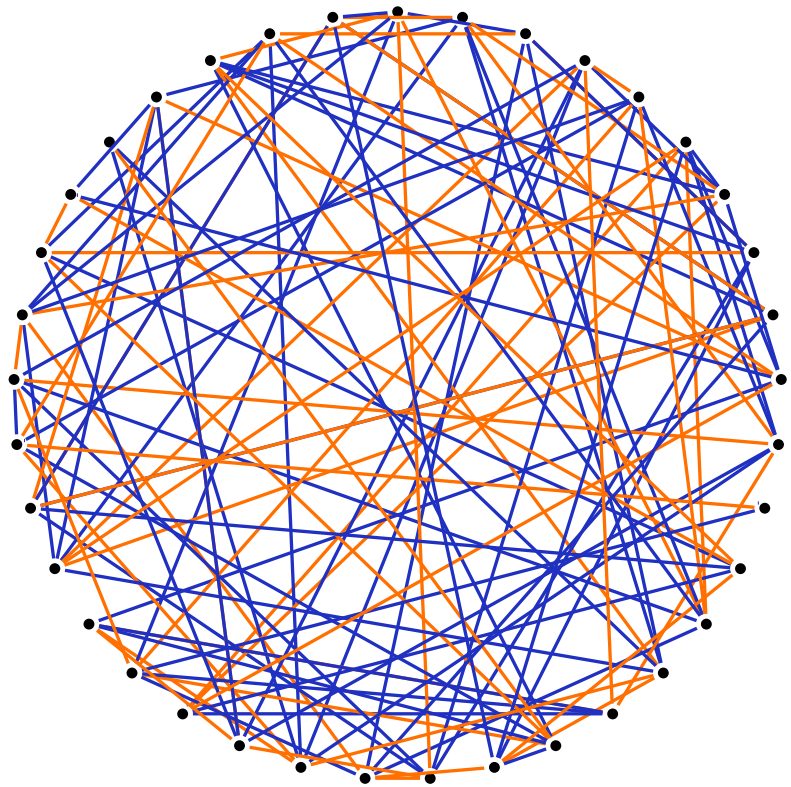
At this time, there are two distinct families of systems:



$q = p$

CSIDH ['siːˌsaɪd]

`https://csidh.isogeny.org`



$q = p^2$

SIDH

`https://sike.org`


CSIDH ['si:ˌsaɪd]
(Castryck, Lange, Martindale, Panny, Renes; 2018)

CSIDH in one slide

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- Choose some small odd primes $\ell_1, \ldots, \ell_n$.
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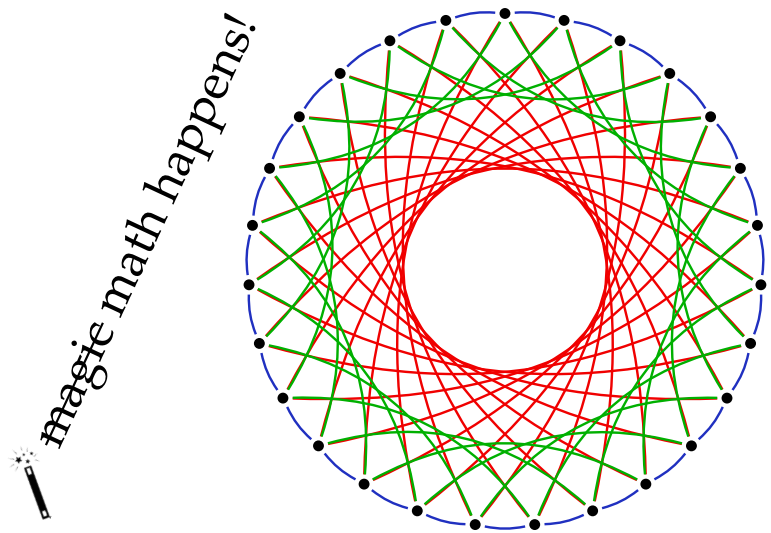

$p = 419$
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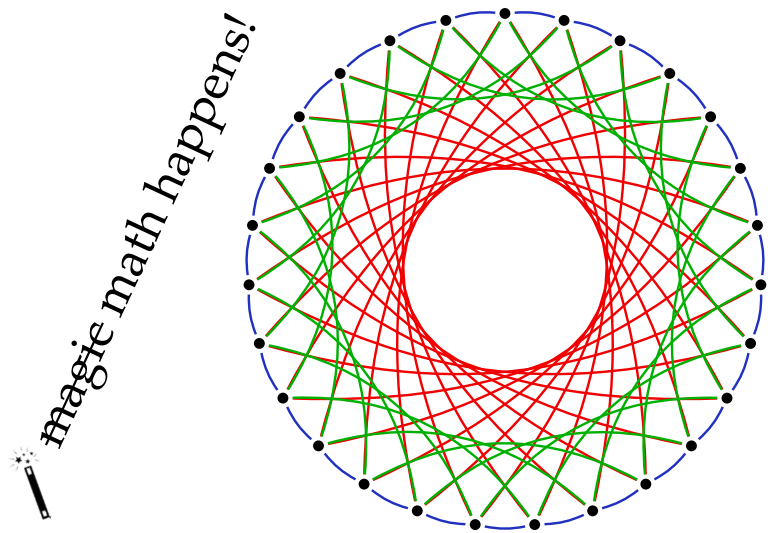

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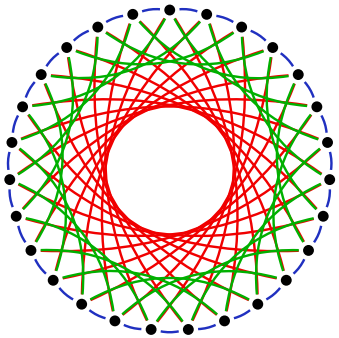
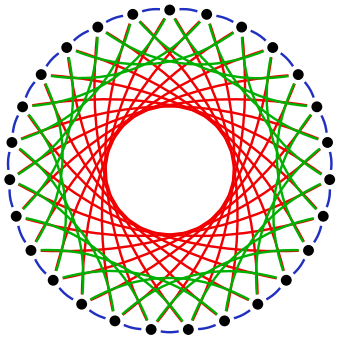
- Walking 'left' and 'right' on any ℓ_i-subgraph is efficient.



Elliptic-curve people may know this graph:
It is the union of depth-0 isogeny volcanoes.



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Typical formulation:

Theorem. Let $\mathcal{O}$ be an imaginary quadratic order and k a field. If the set

$$\mathcal{E}\ell\ell_{\mathcal{O}}(k) = \{\, j(E) \mid E/k \text{ ordinary}, \ \mathrm{End}(E) \cong \mathcal{O} \,\}$$

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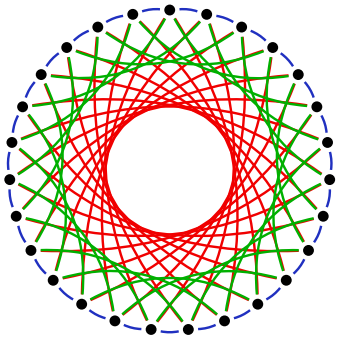

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Less well-known:

This *also* works for supersingular elliptic curves if one restricts to $k = \mathbb{F}_p$, $\cong_{\mathbb{F}_p}$, and $\mathrm{End}_{\mathbb{F}_p}$.

Walking in the CSIDH graph

Supersingular curves have computational benefits:
By taking special p, it is easy to control the group structure!
(Not easy for ordinary curves in 'interesting' cases.)

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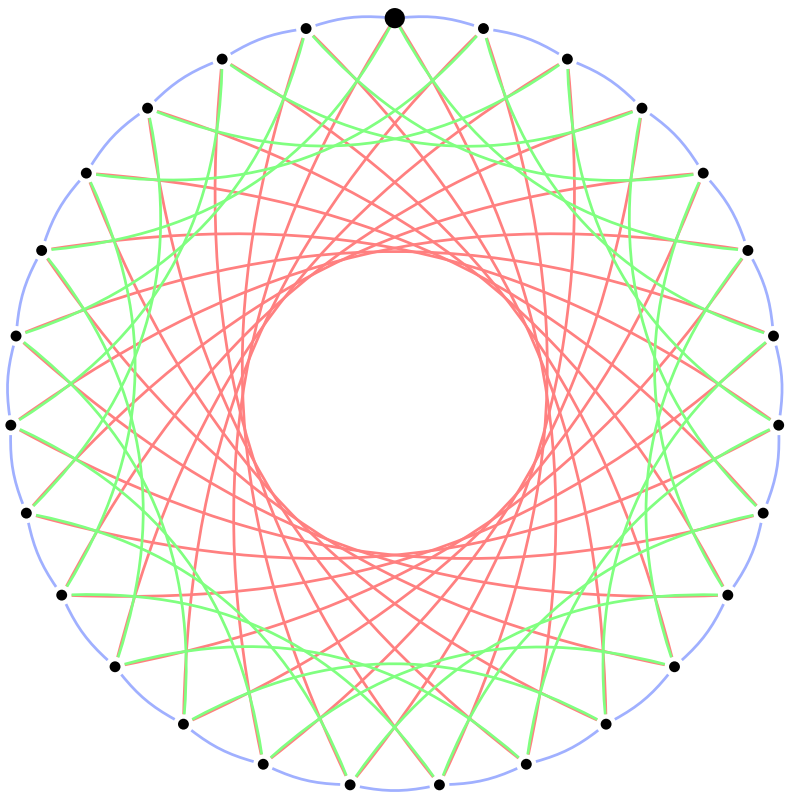
Net result: With x-only arithmetic everything happens over $\mathbb{F}_p$.
$\implies$ Efficient to implement!

CSIDH key exchange

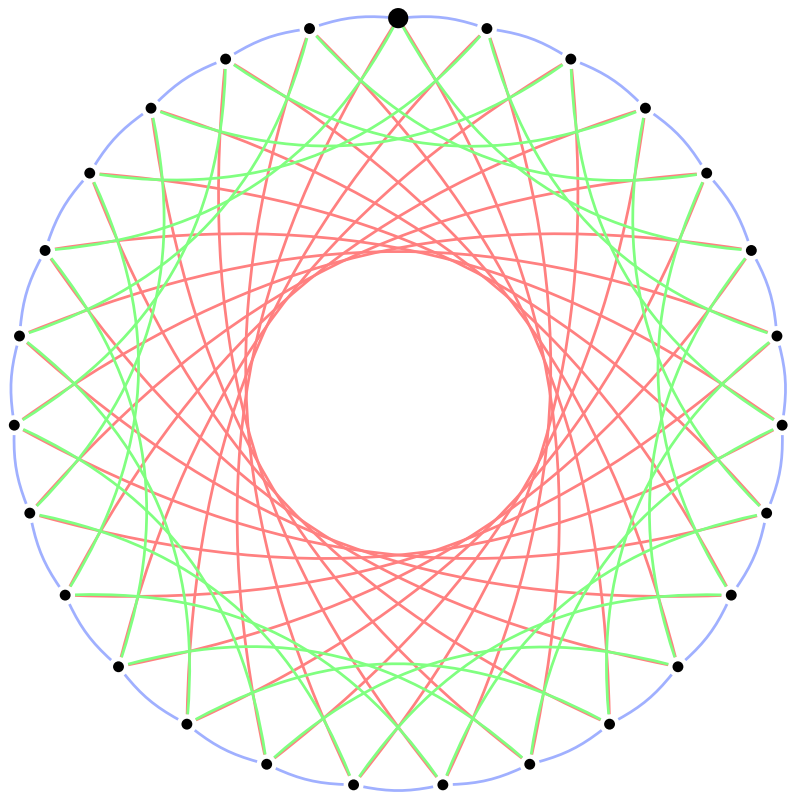
Alice

$[+, +, -, -]$



Bob

$[-, +, -, -]$



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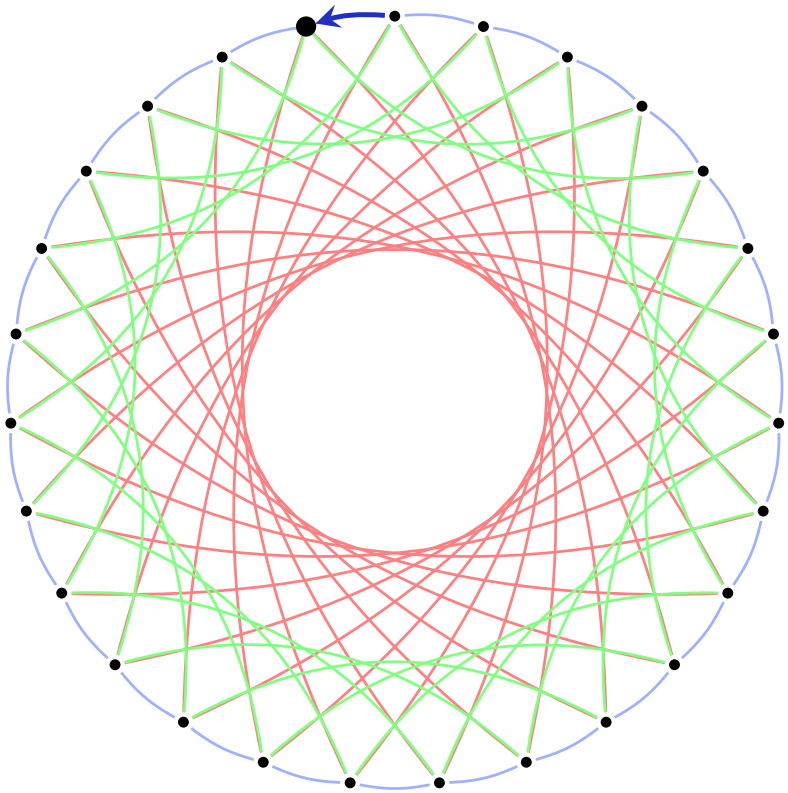

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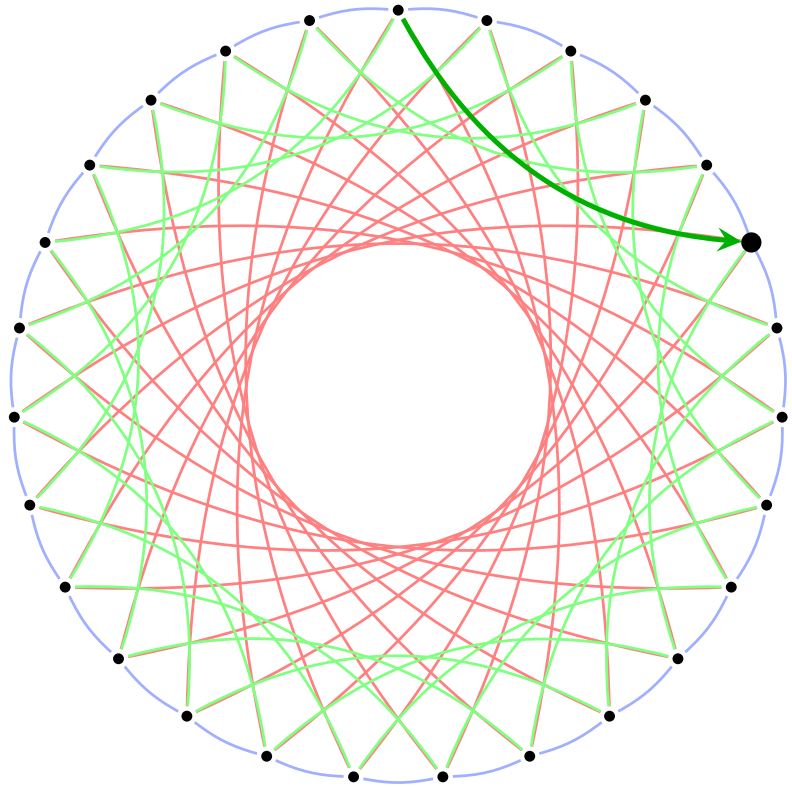

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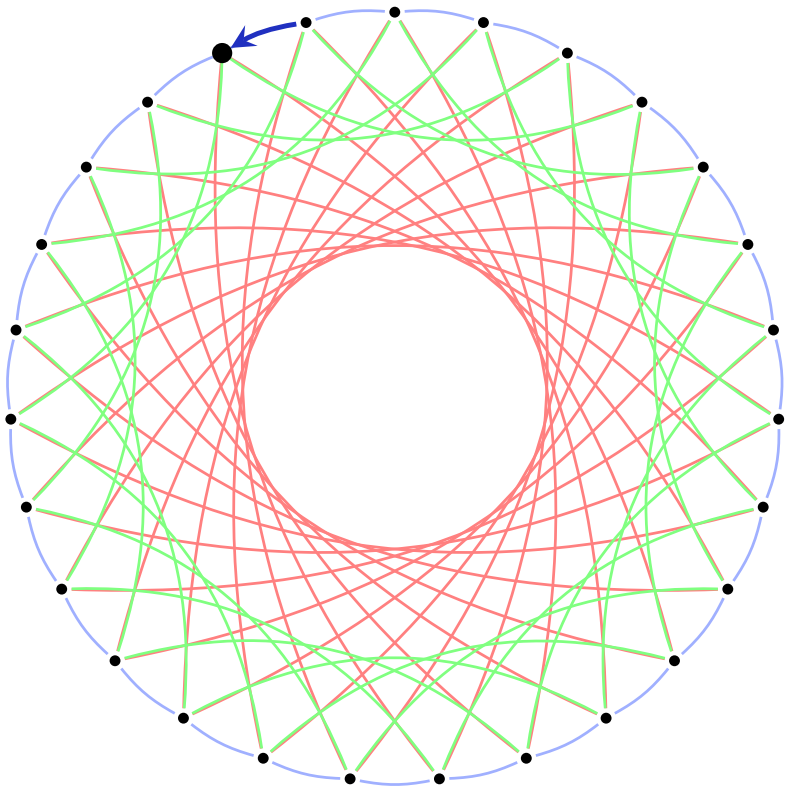

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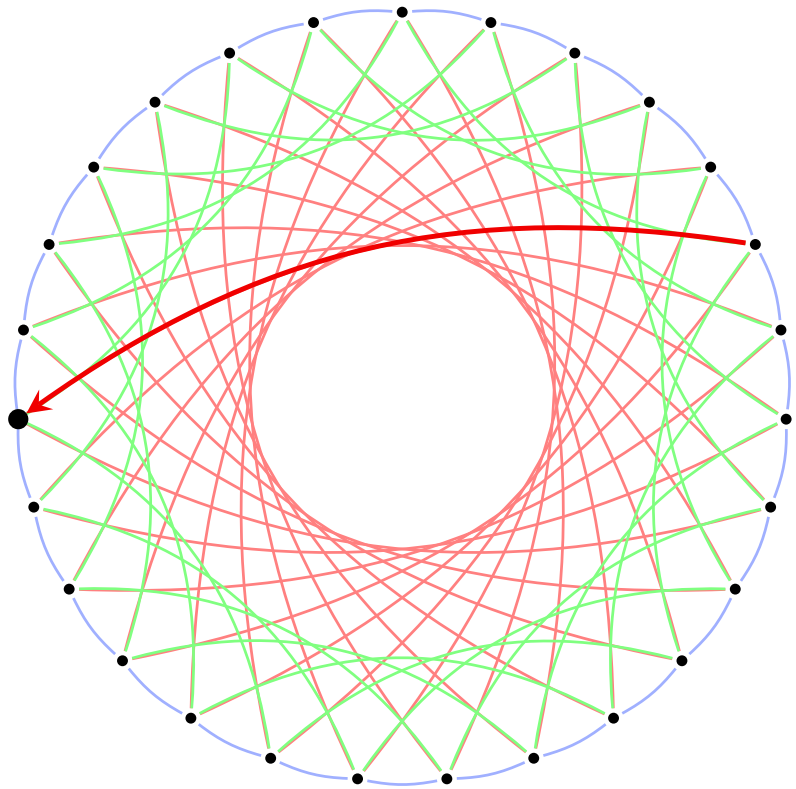

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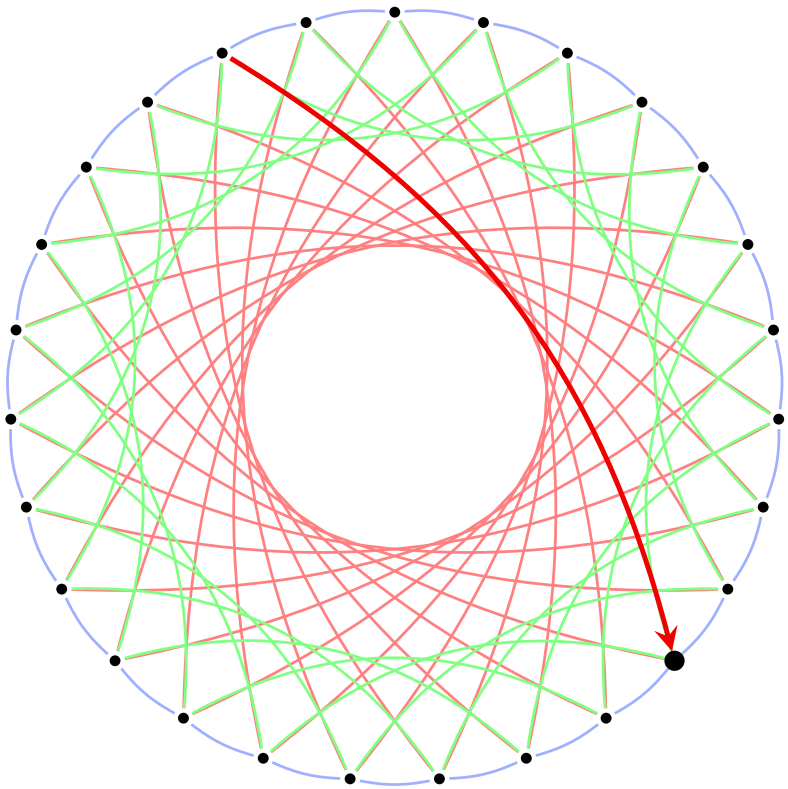

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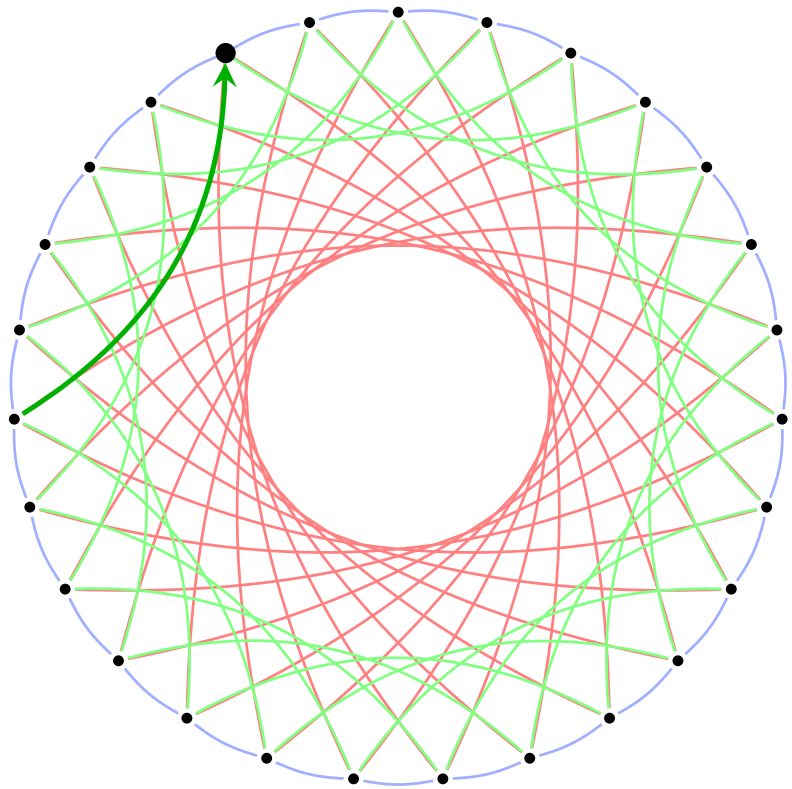

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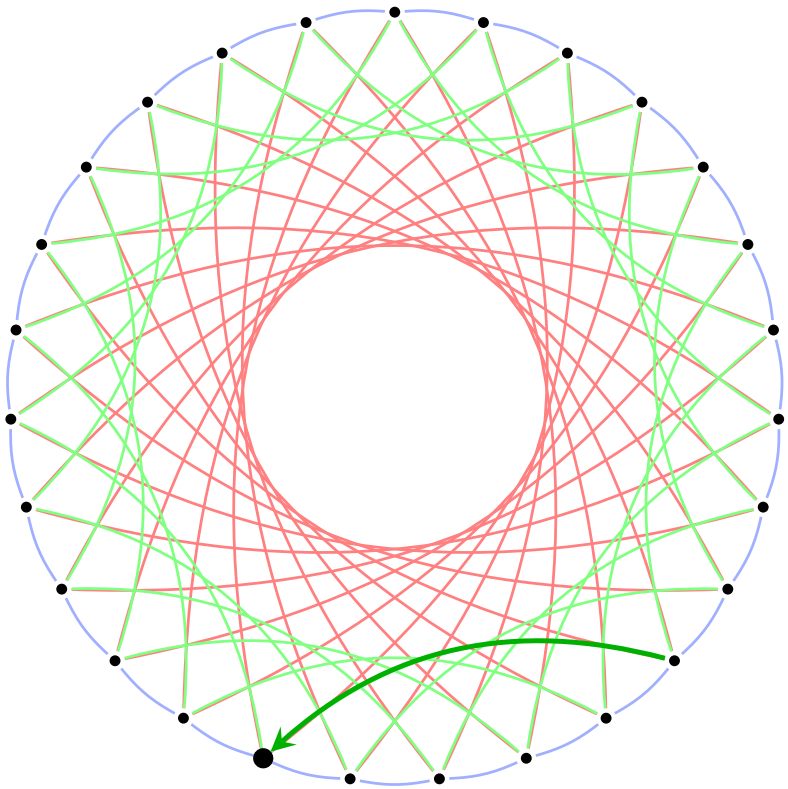

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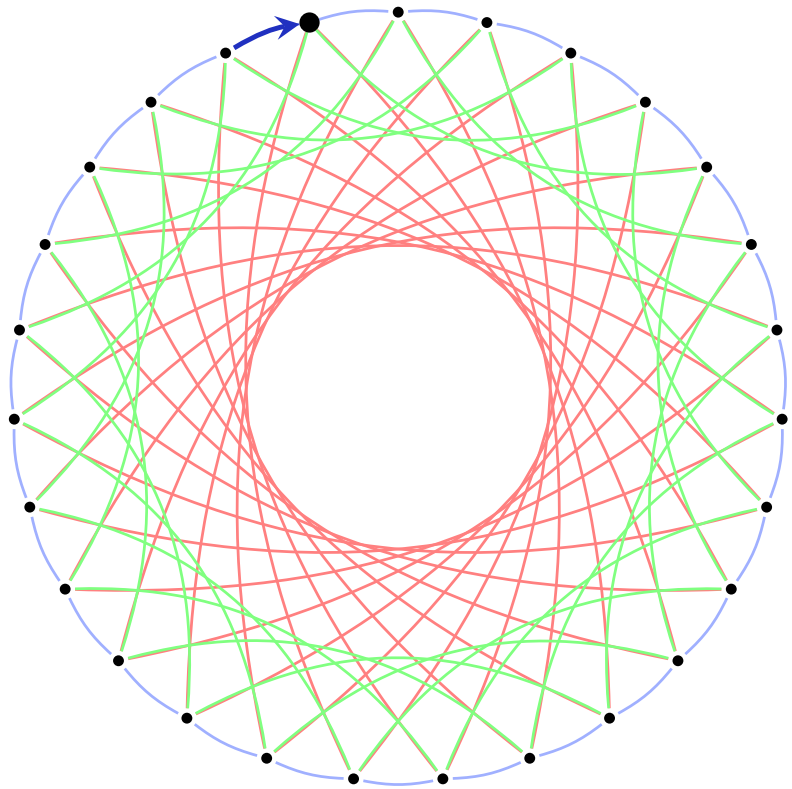

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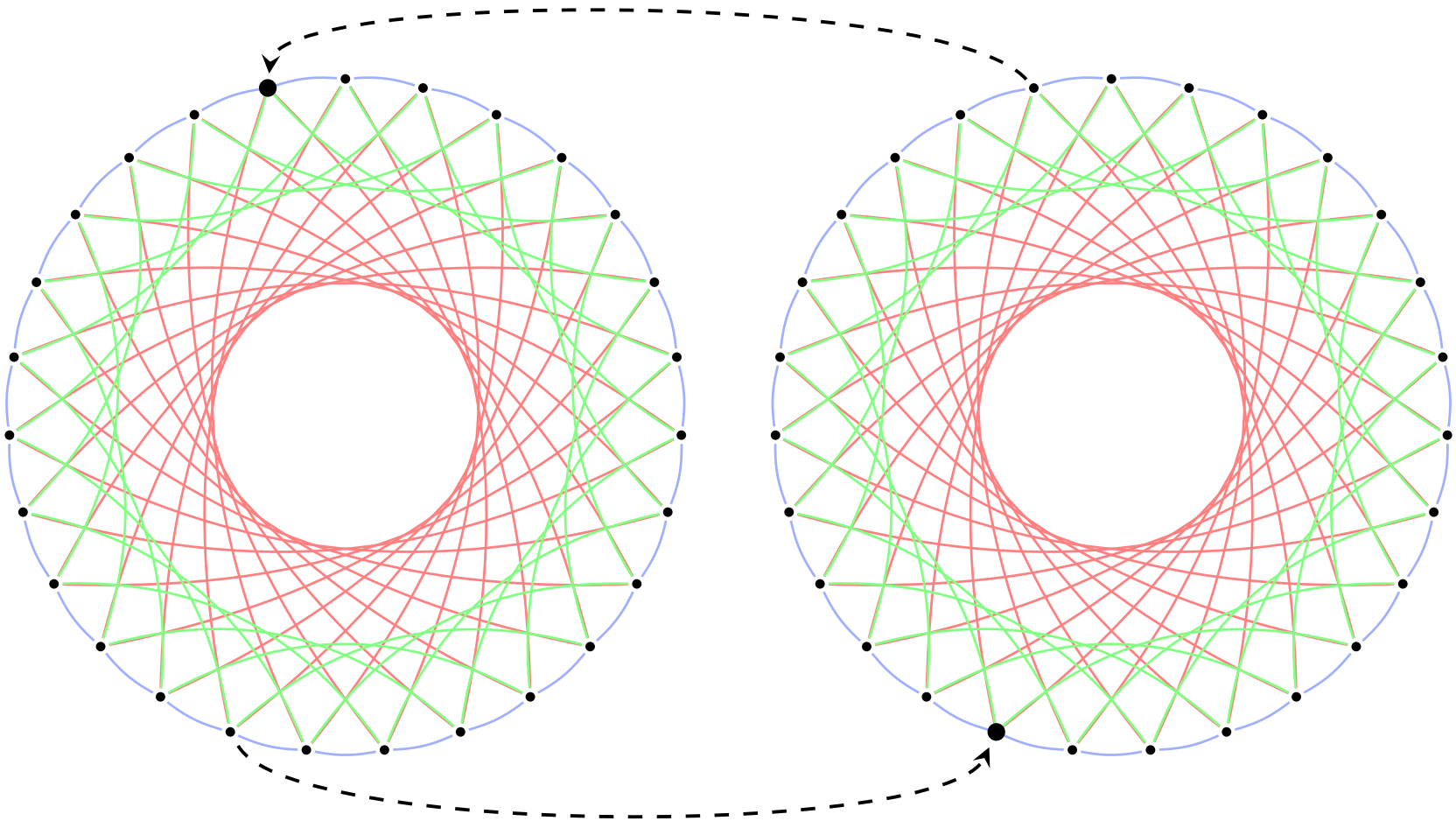

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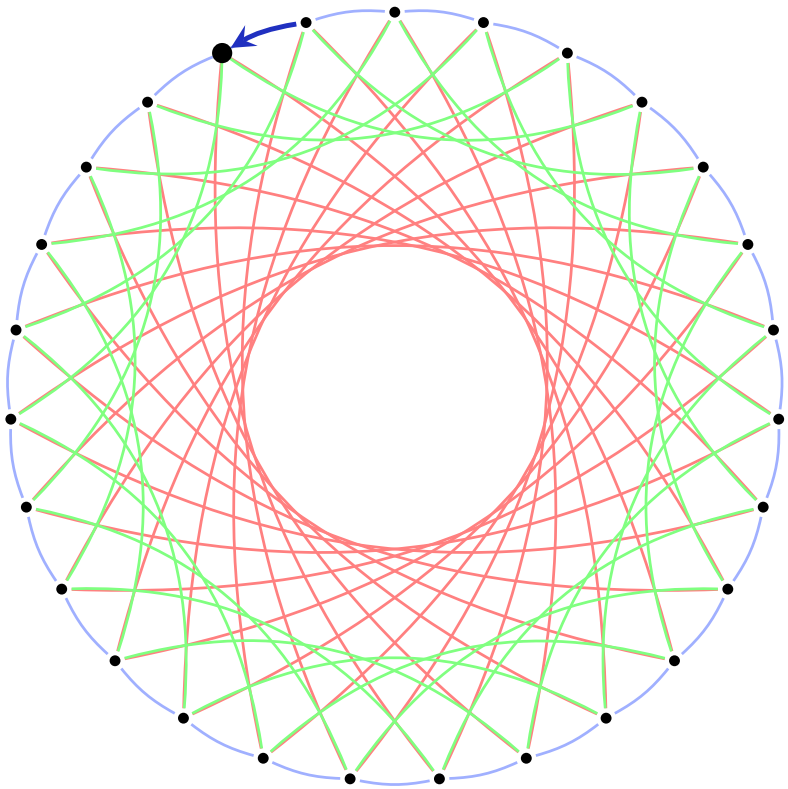

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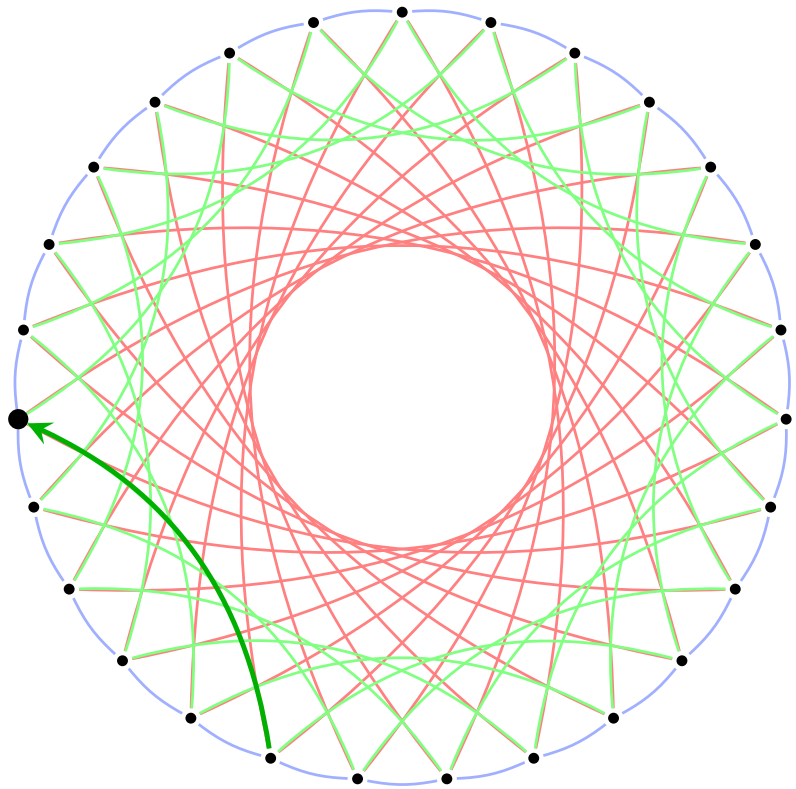

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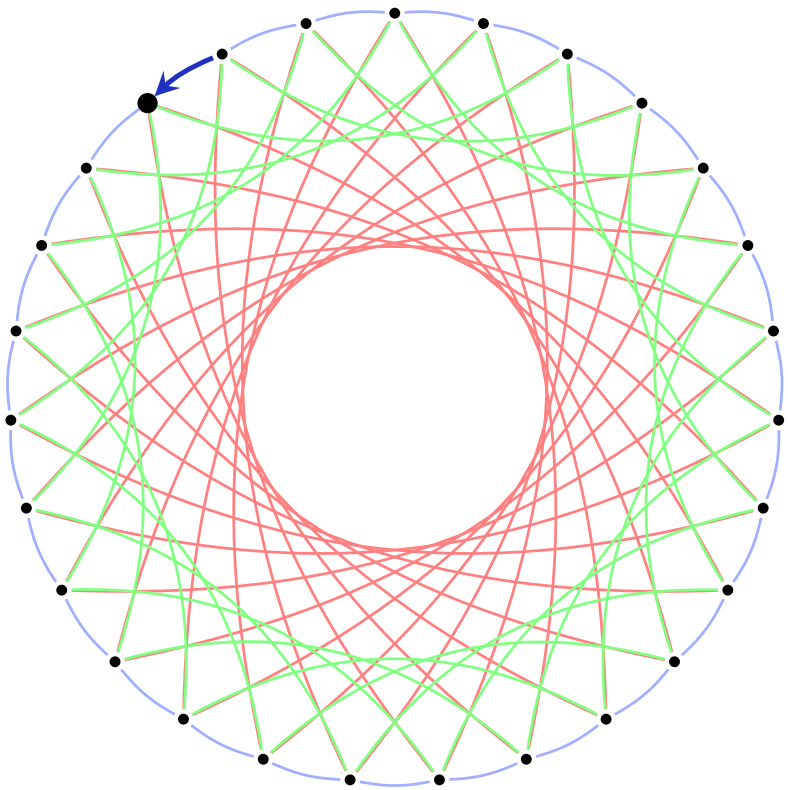

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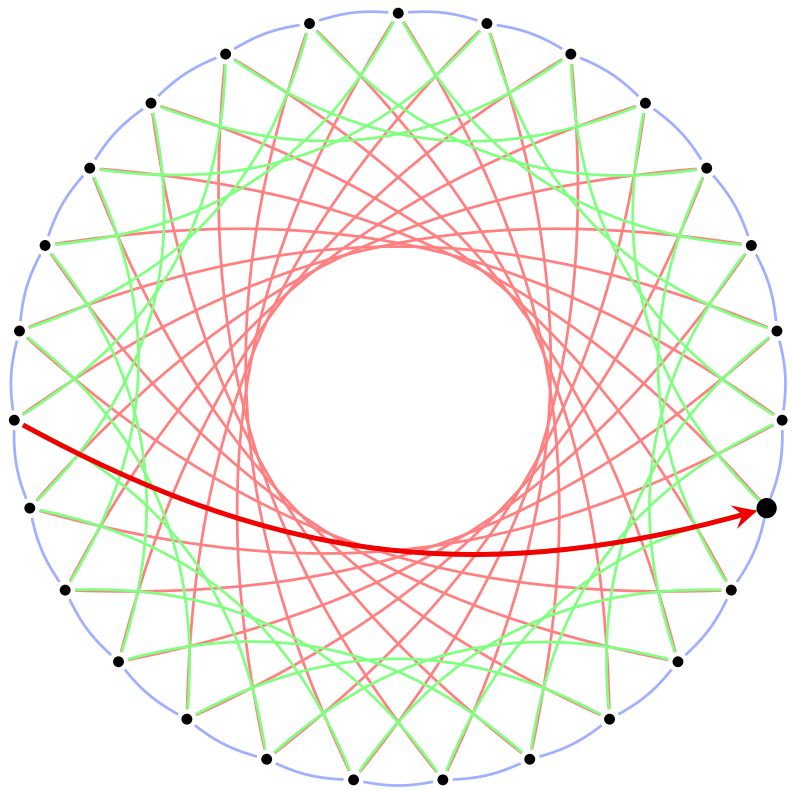

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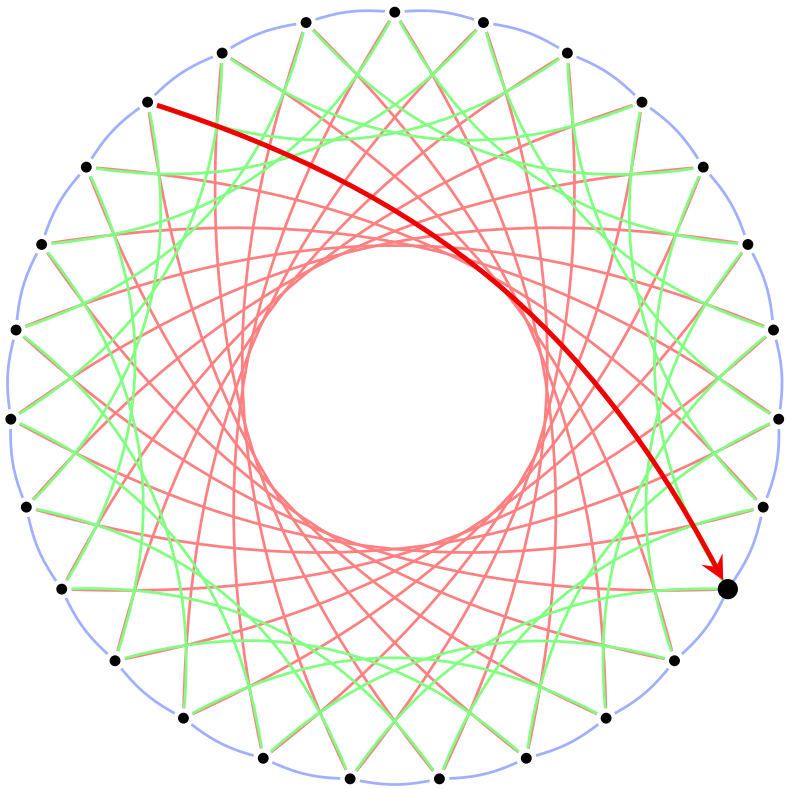

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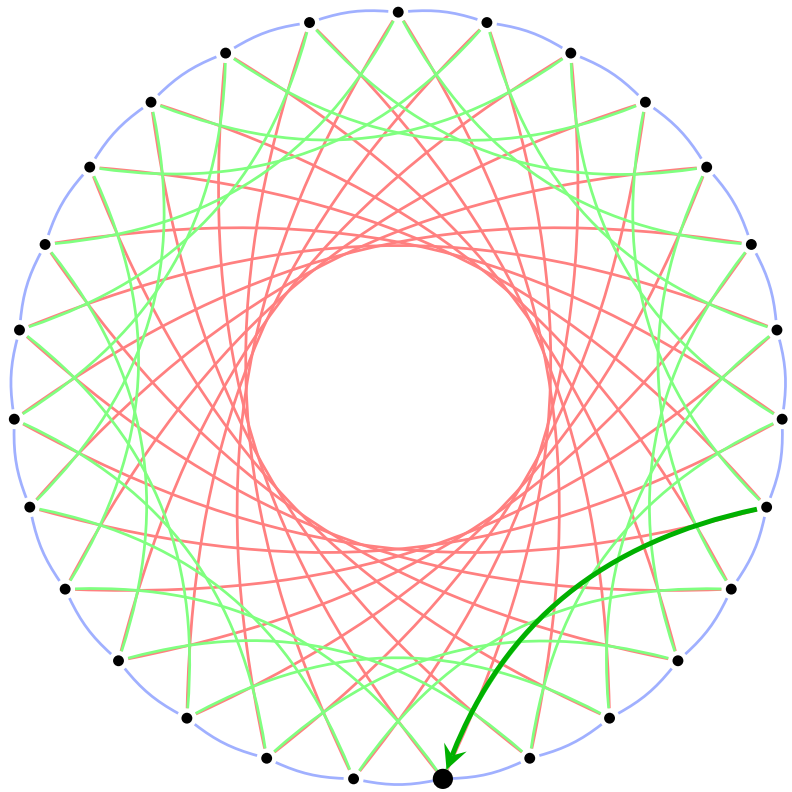

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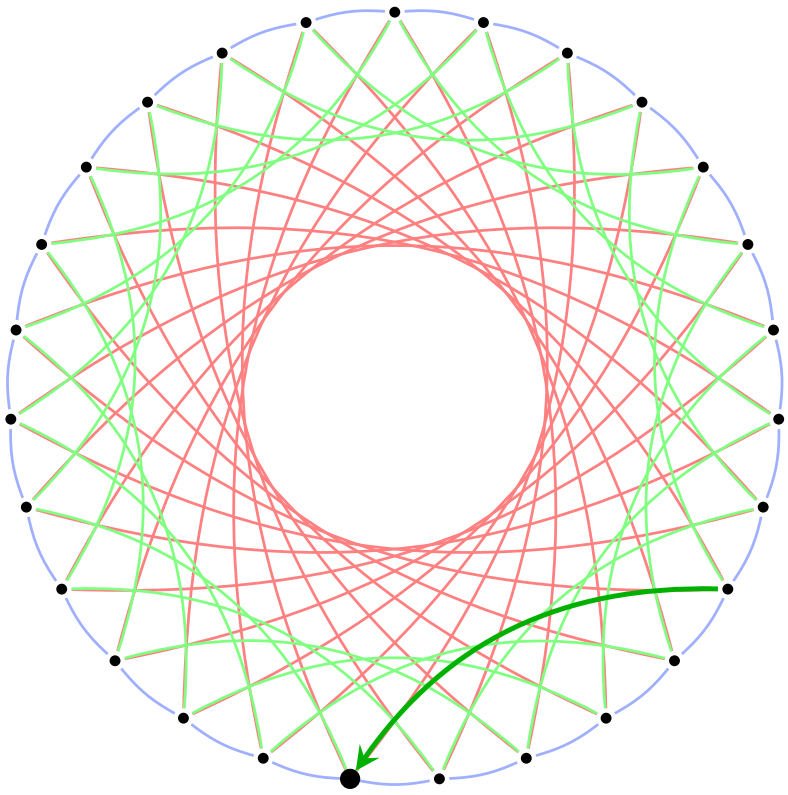

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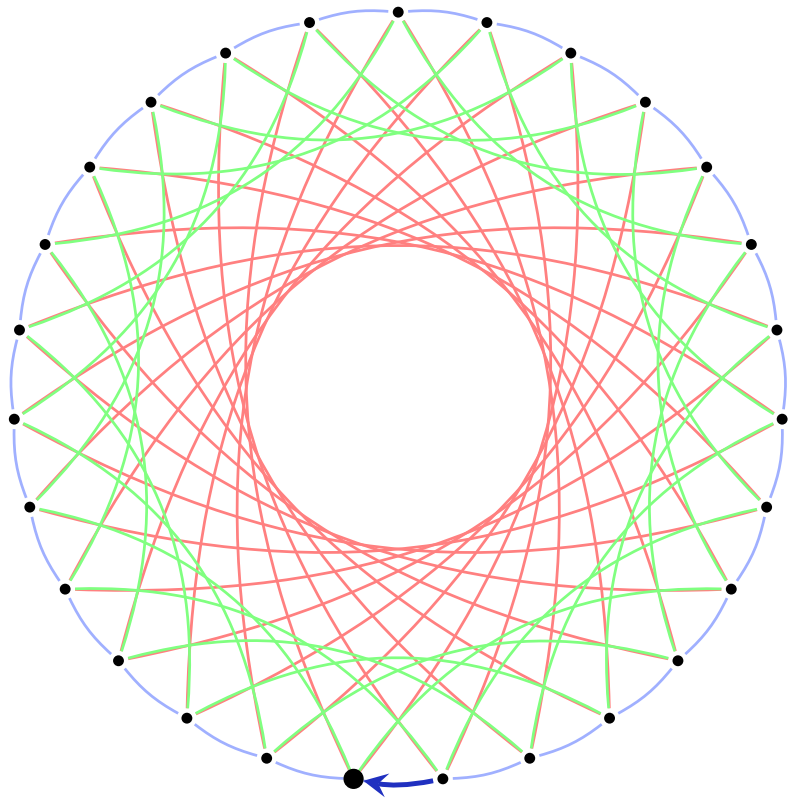

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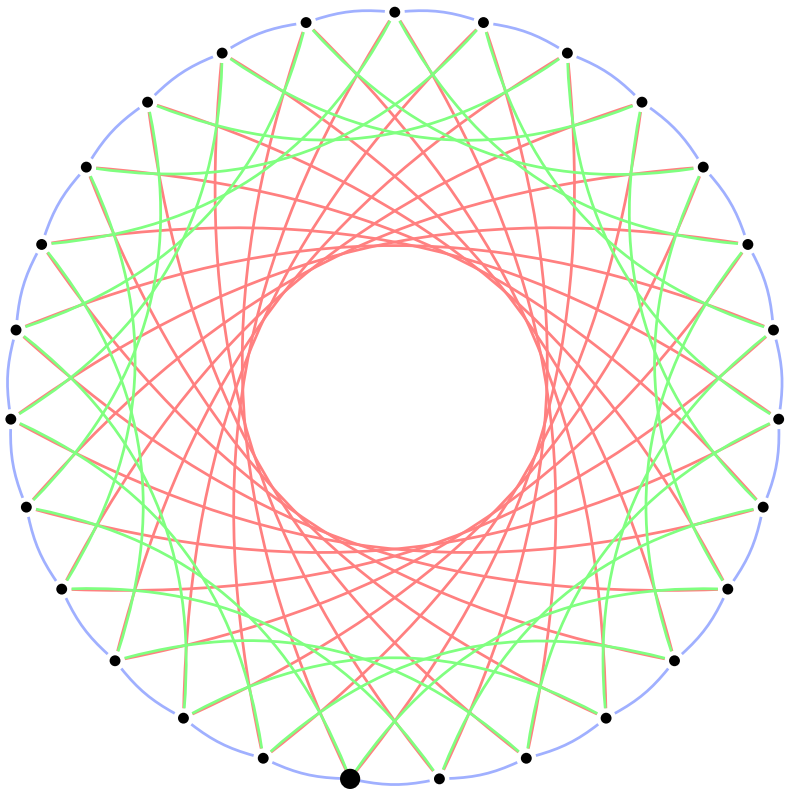

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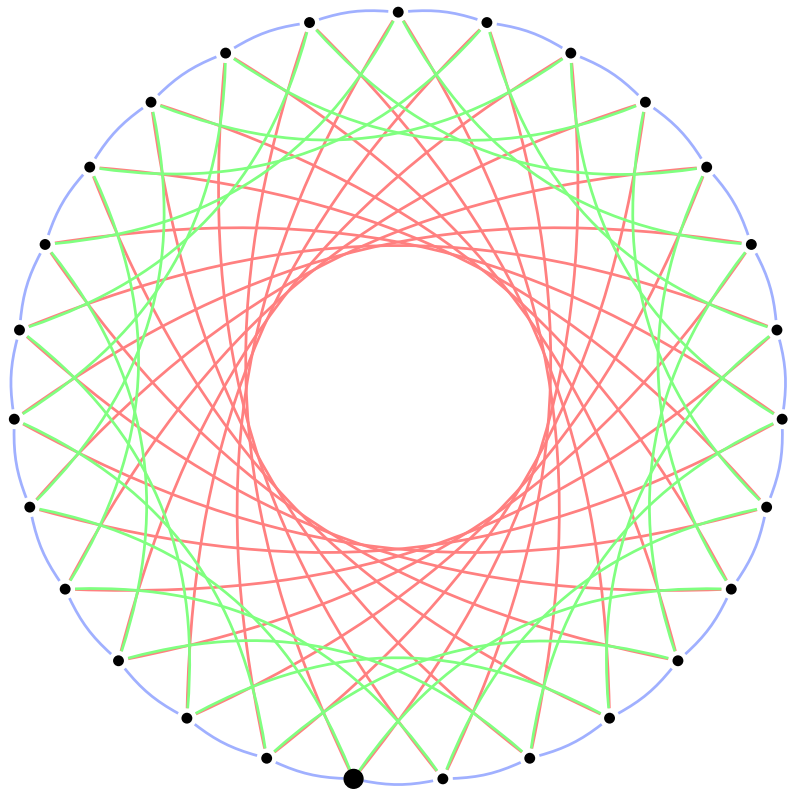
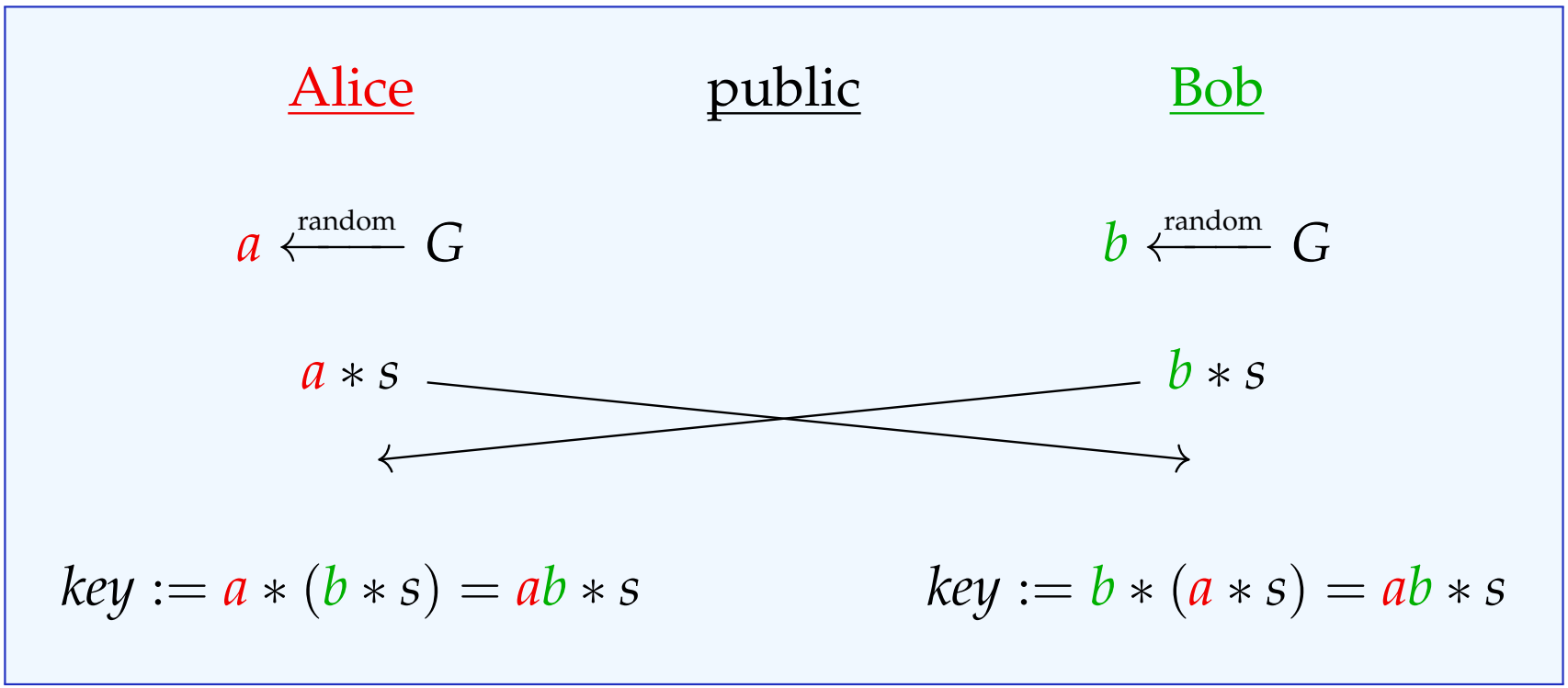


Group=action-based key exchange

Like in the CSIDH example, we *generally* get a key exchange from a commutative group action $G \times S \to S$:



Why no Shor?

Shor computes α from $h = g^\alpha$ by finding the kernel of the map

$$f\colon\ \mathbb{Z}^2 \to G,\ (x, y) \mapsto g^x \underset{\uparrow}{\cdot} h^y$$

For group actions, we generally cannot compose $a * s$ and $b * s$!

Security of CSIDH

Core problem:
Given $E, E' \in X$, find a smooth-degree isogeny $E \to E'$.

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The size of X is $\#\mathrm{cl}(\mathbb{Z}[\sqrt{-p}]) \approx \sqrt{p}$.

$\rightsquigarrow$ best known classical attack: meet-in-the-middle, $\tilde{\mathcal{O}}(p^{1/4})$.

Fully exponential: Complexity $\exp\big((\log p)^{1+o(1)}\big)$.

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Solving abelian hidden shift breaks CSIDH.

$\rightsquigarrow$ non-devastating quantum attack (Kuperberg's algorithm).

Subexponential: Complexity $\exp\big((\log p)^{1/2+o(1)}\big)$. next talk!

Can we avoid Kuperberg's algorithm?

The supersingular isogeny graph over $\mathbb{F}_{p^2}$ has less structure.

- **SIDH** uses the full $\mathbb{F}_{p^2}$-isogeny graph. No group action!

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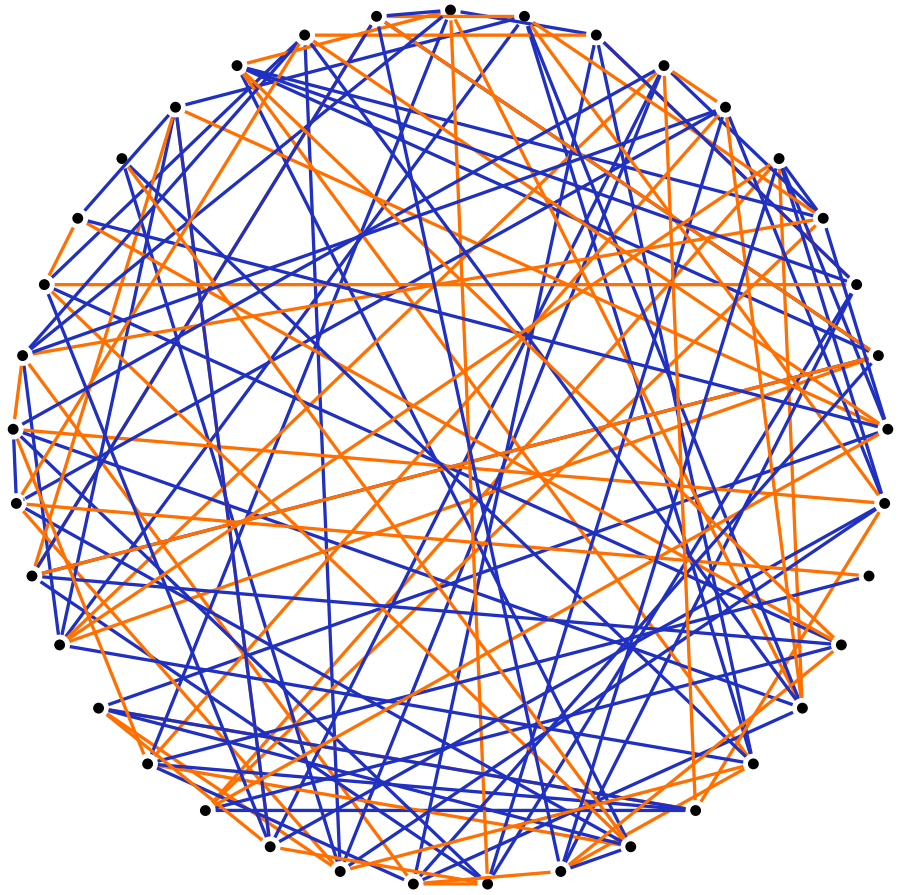
- Problem: also no more intrinsic sense of direction.

  "It all bloody looks the same!" — a famous isogeny cryptographer

$\rightsquigarrow$ need extra information to let Alice & Bob's walks commute.



Now: SIDH (Jao, De Feo; 2011)

(...whose name doesn't allow for nice pictures of beaches...)

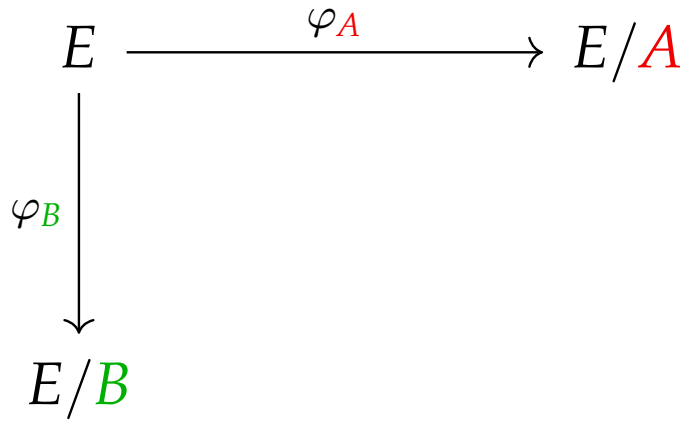
SIDH: High-level view

E

SIDH: High-level view

- Alice & Bob pick secret subgroups A and B of E.

SIDH: High-level view



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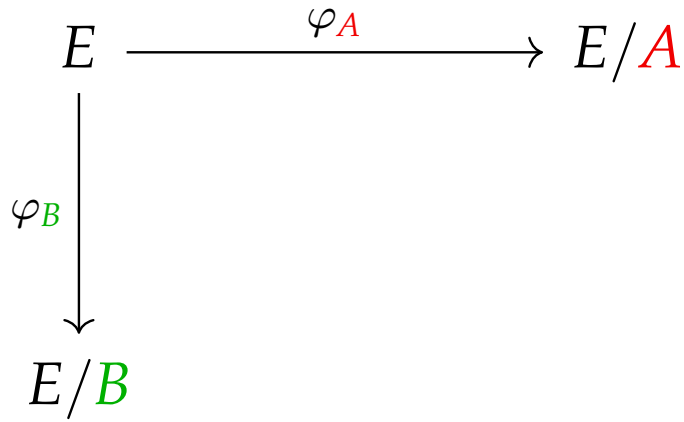

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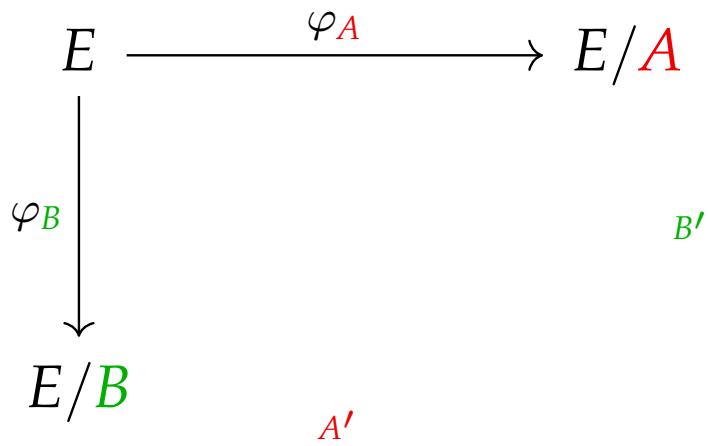

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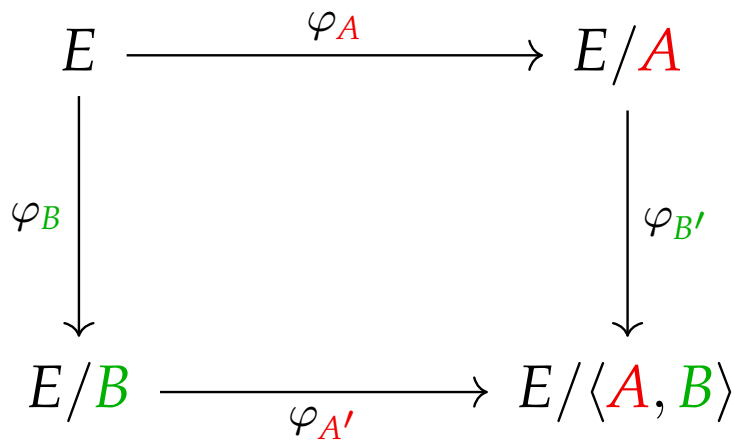

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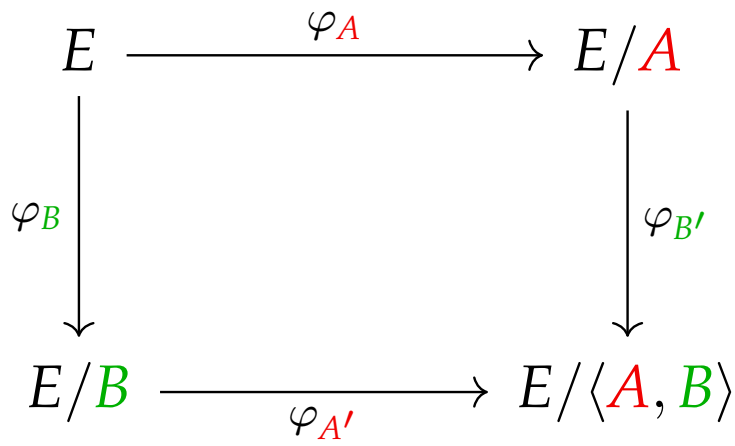
- They both compute the shared secret
  $$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

SIDH: High-level view



- Alice & Bob pick secret subgroups A and B of E.
- Alice computes $\varphi_A : E \to E/A$; Bob computes $\varphi_B : E \to E/B$.
  (These isogenies correspond to walking on the isogeny graph.)
- Alice and Bob transmit the values E/A and E/B.
- Alice somehow obtains $A' := \varphi_B(A)$. (Similar for Bob.)
- They both compute the shared secret
  $$(E/B)/A' \cong E/\langle A, B\rangle \cong (E/A)/B'.$$

By the way: This is *also* the high-level view for CSIDH... 🙃

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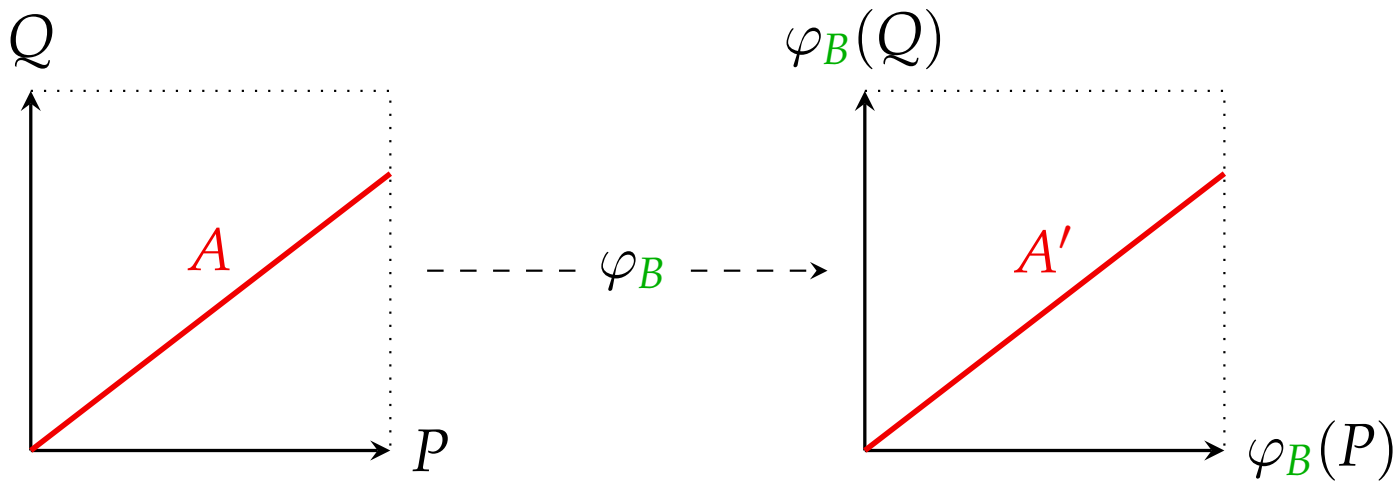
$\Longrightarrow$ Now Alice can compute A' as $\langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$!



SIDH in one slide

Public parameters:

- a large prime $p = 2^n 3^m - 1$ and a supersingular $E/\mathbb{F}_p$
- bases (P, Q) and (R, S) of $E[2^n]$ and $E[3^m]$ (recall $E[k] \cong \mathbb{Z}/k \times \mathbb{Z}/k$)

Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \dots 2^n - 1\}$		$b \xleftarrow{\text{random}} \{0 \dots 3^m - 1\}$
$A := \langle P + [a]Q \rangle$		$B := \langle R + [b]S \rangle$
compute $\varphi_A : E \to E/A$		compute $\varphi_B : E \to E/B$
$E/A,\ \varphi_A(R),\ \varphi_A(S)$	$\longrightarrow$	
	$\longleftarrow$	$E/B,\ \varphi_B(P),\ \varphi_B(Q)$
$A' := \langle \varphi_B(P) + [a]\varphi_B(Q) \rangle$		$B' := \langle \varphi_A(R) + [b]\varphi_A(S) \rangle$
$s := j\big((E/B)/A'\big)$		$s := j\big((E/A)/B'\big)$

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Bottom line: Fully exponential. Complexity $\exp\left((\log p)^{1+o(1)}\right)$.

Fragen?

Questions?

Domande?

Dumondas?