Isogeny-based Cryptography

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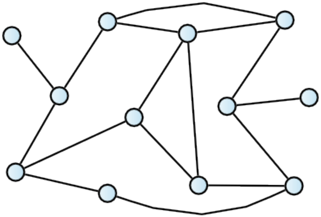
Big picture

- Isogenies are a source of exponentially-sized graphs.



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  That is: some *well-behaved* “directions” to describe paths.

Finding graphs with *almost* all of these properties is easy —
but getting all at once seems rare.

Crypto on graphs?

Diffie–Hellman key exchange 1976

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today elliptic curves)
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Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \dots q-1\}$		$b \xleftarrow{\text{random}} \{0 \dots q-1\}$
g^a		g^b
$s := (g^b)^a$		$s := (g^a)^b$

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Fundamental reason this works: $\cdot^a$ and $\cdot^b$ are commutative!

Diffie–Hellman: Bob vs. Eve

Bob

1. Set $t \leftarrow g$.
2. Set $t \leftarrow t \cdot g$.
3. Set $t \leftarrow t \cdot g$.
4. Set $t \leftarrow t \cdot g$.

...

$b-2$. Set $t \leftarrow t \cdot g$.

$b-1$. Set $t \leftarrow t \cdot g$.

b. Publish $B \leftarrow t \cdot g$.

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Is this a good idea?

Diffie–Hellman: Bob vs. Eve

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...

$b{-}2$. Set $t \leftarrow t \cdot g$.

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b. Publish $B \leftarrow t \cdot g$.

<u>**Attacker** Eve</u>

1. Set $t \leftarrow g$. If $t = B$ return 1.
2. Set $t \leftarrow t \cdot g$. If $t = B$ return 2.
3. Set $t \leftarrow t \cdot g$. If $t = B$ return 3.
4. Set $t \leftarrow t \cdot g$. If $t = B$ return 3.

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$b{-}2$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b{-}2$.

$b{-}1$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b{-}1$.

b. Set $t \leftarrow t \cdot g$. If $t = B$ return b.

$b{+}1$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b + 1$.

$b{+}2$. Set $t \leftarrow t \cdot g$. If $t = B$ return $b + 2$.

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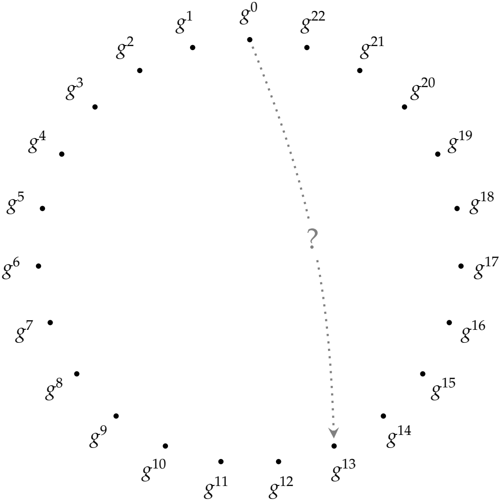
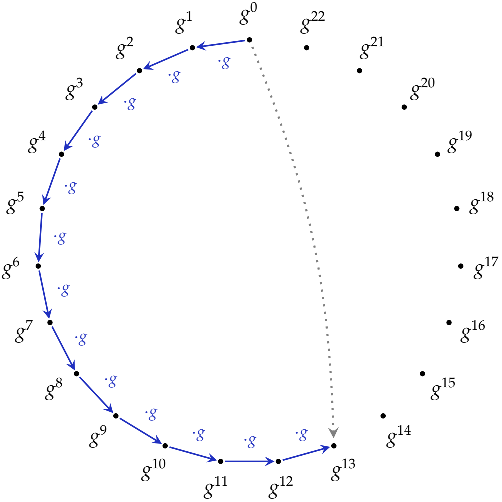
Effort for both: $O(\#G)$. Bob needs to be smarter.

(This attacker is also kind of dumb, but that doesn't matter for my point here.)



Bob computes his public key g^{13} from g.

multiply



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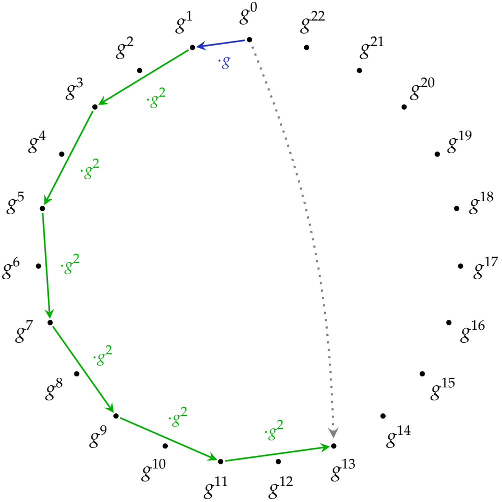
Square-and-multiply



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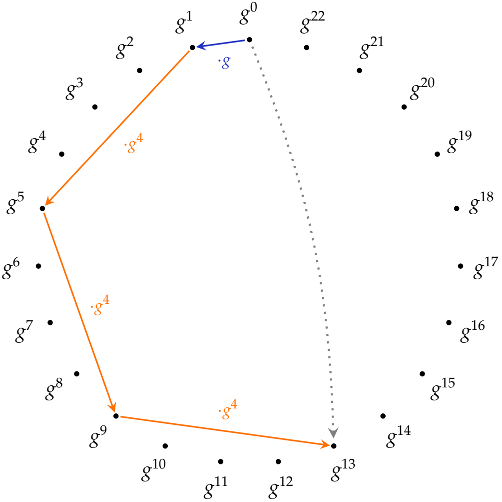
Square-and-multiply-and-square-and-multiply



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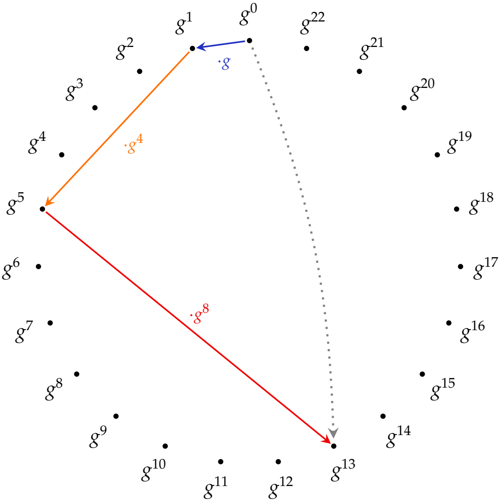

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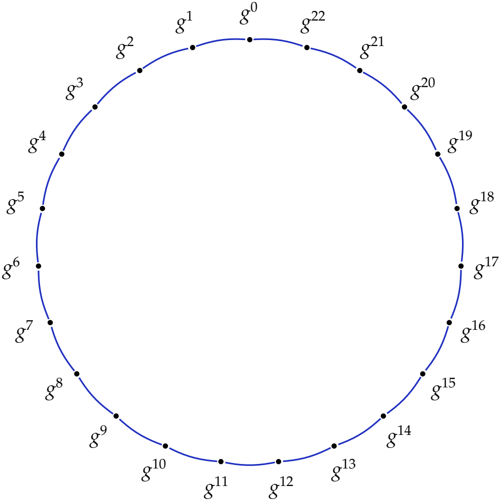
Square-and-multiply as graphs



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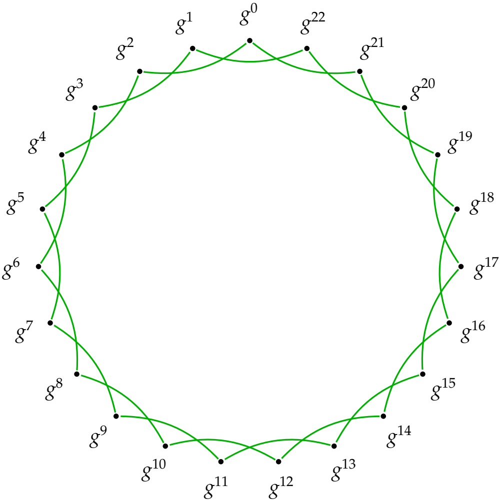

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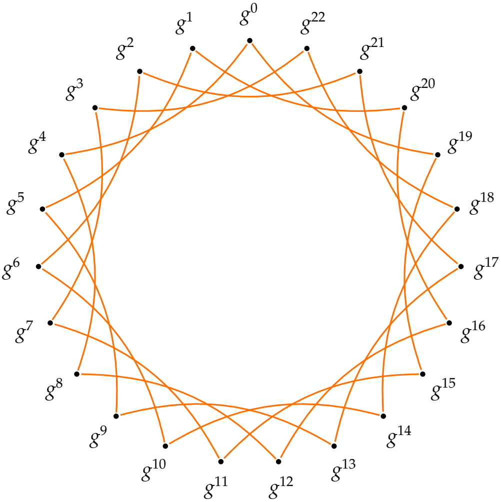

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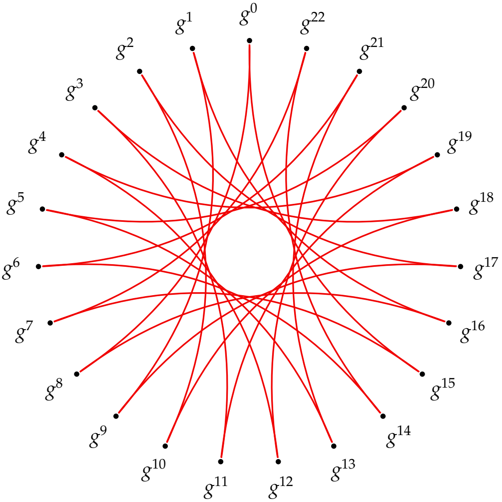
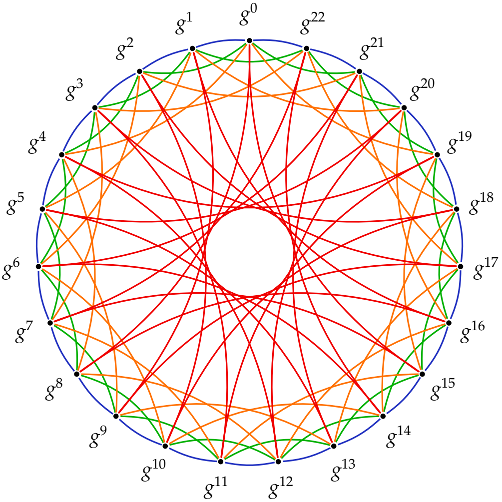


Square-and-multiply as a graph



Crypto on graphs?

We’ve been doing it all the time!

The fast mixing requirement

Fast mixing: paths of length $\log(\#\,\text{nodes})$ to everywhere.

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Shor's quantum algorithm computes α from g^α in any group in polynomial time.

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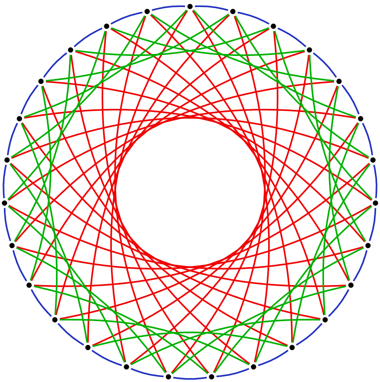
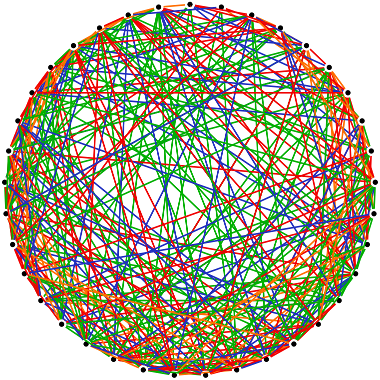
The beauty and the beast

Components of particular isogeny graphs look like this:





Which of these is good for crypto?

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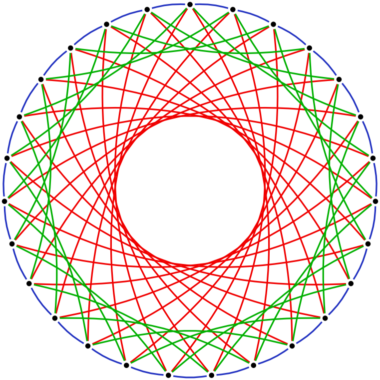

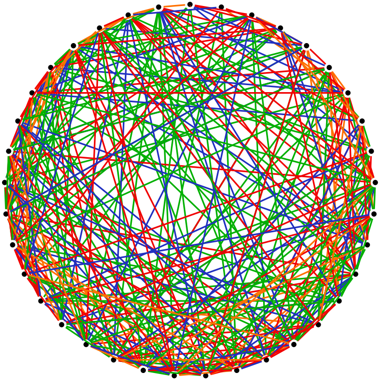

Which of these is good for crypto? **Both.** ⌣̈

Plan for this lecture

- High-level overview for intuition. ✓
- Elliptic curves & isogenies.
- The CGL hash function.
- The CSIDH non-interactive key exchange.
- Hardness of isogeny problems, and reductions.
- The SQIsign signature scheme.
- Transcending to higher dimensions.

Stand back!

We're going to do math.

Elliptic curves

An elliptic curve over a field F of characteristic $\notin \{2, 3\}$ is* an equation of the form

$$E\colon\ y^2 = x^3 + ax + b$$

with $a, b \in F$ such that $4a^3 + 27b^2 \neq 0$.

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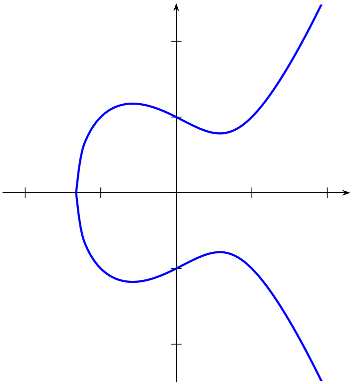
- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

do **not** remember these formulas!

$$\left(\lambda^2 - x_1 - x_2,\ \lambda(2x_1 + x_2 - \lambda^2) - y_1\right)$$

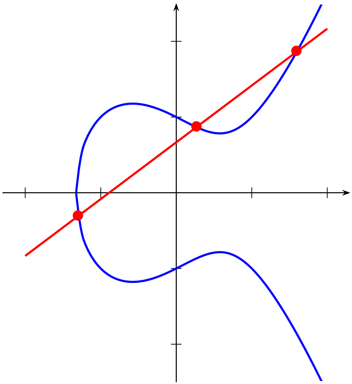
where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + a}{2y_1}$ otherwise.

Elliptic curves (picture over $\mathbb{R}$)



The elliptic curve $y^2 = x^3 - x + 1$ over $\mathbb{R}$.

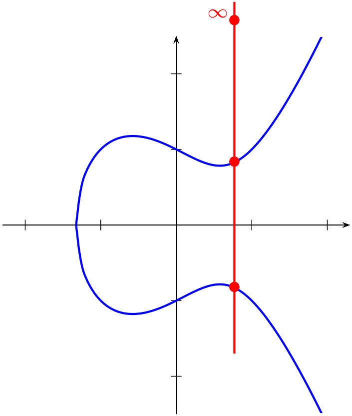
Elliptic curves (picture over $\mathbb{R}$)



Addition law:

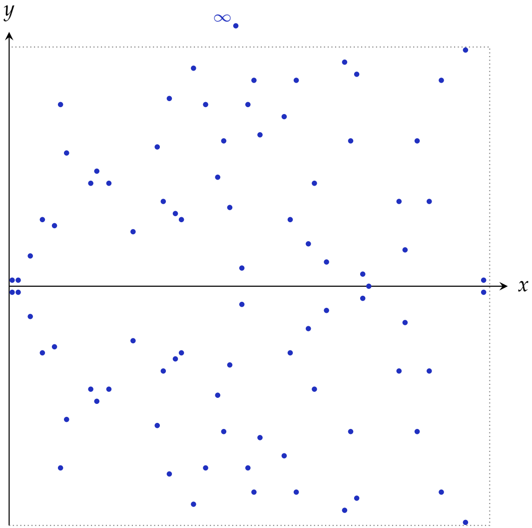
$$P + Q + R = \infty \iff \{P, Q, R\} \text{ on a straight line.}$$

Elliptic curves (picture over $\mathbb{R}$)



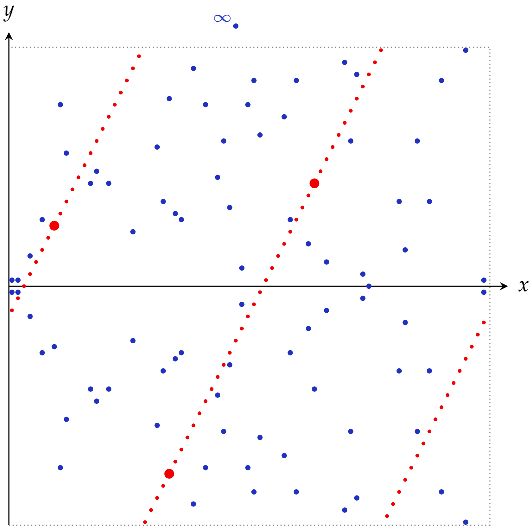
The *point at infinity* ∞ lies on every vertical line.

Elliptic curves (picture over $\mathbb{F}_p$)



The same curve $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

Elliptic curves (picture over $\mathbb{F}_p$)



The addition law of $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

In SageMath:

```
sage: E = EllipticCurve(GF(101), [5,6,7,8,9])
sage: E
Elliptic Curve defined by
            y^2 + 5*x*y + 7*y = x^3 + 6*x^2 + 8*x + 9
            over Finite Field of size 101
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sage: P = E(3, 18)       # constructing points
sage: Q = E(8, 75)
sage: P + Q              # point addition
(73 : 24 : 1)
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(73 : 24 : 1)
sage: P - P
(0 : 1 : 0)              # point at infinity
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Public parameters:
an elliptic curve E and a point $P \in E$ of large prime order ℓ.

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Define scalar multiplication $[n]P := \underbrace{P + \cdots + P}_{n \text{ times}}$. (Use double-and-add!)

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Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \ldots \ell{-}1\}$		$b \xleftarrow{\text{random}} \{0 \ldots \ell{-}1\}$
$[a]P$		$[b]P$
$s := [a]([b]P)$	$\longleftarrow$ equal! $\longrightarrow$	$s := [b]([a]P)$

Fields of definition

Generally, things can be defined over extension fields:
For example, $(0, \sqrt{-1})$ is a point of $y^2 = x^3 - 1$.

Let k be a field.

An elliptic curve/point/isogeny is defined over k or k-rational if the coefficients in its equation/formula lie in k.

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For E/k, write $E(k)$ for the set of points of E defined over k.

Note: Simply writing E means $E(\bar{k})$, i.e., points over *all* extension fields.

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```
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Elliptic Curve defined by y^2 = x^3 + 5*x^2 + x
                over Finite Field of size 101
sage: F.<t> = GF(101^2)
sage: E(11, 69*t + 64)
ValueError: 69*t + 64 is not in the image of #...
sage: EE = E.change_ring(F)
sage: EE(11, 69*t + 64)
(11 : 69*t + 64 : 1)
```

Isogenies

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...are just fancily-named

nice maps

between elliptic curves.

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An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

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Reminder:

A rational function is $f(x, y)/g(x, y)$ where f, g are polynomials.

A group homomorphism φ satisfies $\varphi(P + Q) = \varphi(P) + \varphi(Q)$.

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A group homomorphism φ satisfies $\varphi(P + Q) = \varphi(P) + \varphi(Q)$.

The kernel of an isogeny $\varphi\colon E \to E'$ is $\{P \in E \,:\, \varphi(P) = \infty\}$.
The degree of a separable* isogeny is the size of its kernel.

Isogenies (examples)

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Example #1: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

Isogenies (examples)

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Example #2: For any a and b, the map $\iota\colon (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}.$$

It is an *isomorphism*; its kernel is $\{\infty\}$.

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Example #3: For each $m \neq 0$, the multiplication-by-m map

$$[m] \colon E \to E$$

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Example #3: For each $m \neq 0$, the multiplication-by-m map

$$[m] \colon E \to E$$

is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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Example #4: For $E/\mathbb{F}_q$, the map

$$\pi\colon (x, y) \mapsto (x^q, y^q)$$

is a degree-q isogeny, the *Frobenius endomorphism*.

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The kernel of $\pi - 1$ is precisely the set of rational points $E(\mathbb{F}_q)$.
Important fact: An isogeny φ is $\mathbb{F}_q$-rational iff $\pi \circ \varphi = \varphi \circ \pi$.

In SageMath:

```
sage: E = EllipticCurve(GF(101), [1,0])
sage: mu = E.scalar_multiplication(5)
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In SageMath:

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Scalar-multiplication endomorphism [5]
    of Elliptic Curve defined by y^2 = x^3 + x
    over Finite Field of size 101
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    over Finite Field of size 101
sage: mu.rational_maps()
((x^25 + x^23 + ... + 14*x^3 + 25*x)
    /(25*x^24 + 14*x^22 - ... + x^2 + 1),
 (50*x^36*y + 20*x^34*y + ... + 45*x^2*y + 48*y)
    /(-12*x^36 - 2*x^34 + ... - 26*x^2 + 50))
```

The isogeny relation

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<u>Tate's theorem</u>:
$E, E'/\mathbb{F}_q$ are isogenous over $\mathbb{F}_q$ if and only if $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$.

(The Schoof–Elkies–Atkin algorithm can compute $\#E(\mathbb{F}_q)$ efficiently!)

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(The Schoof–Elkies–Atkin algorithm can compute $\#E(\mathbb{F}_q)$ efficiently!)

$\Longrightarrow$ Bottom line: Being isogenous is an equivalence relation.
Over finite fields, we can easily test it.

Isogenies and kernels

For any finite subgroup G of E, there exists a unique[1] separable* isogeny $\varphi_G \colon E \to E'$ with kernel G.

[1](up to isomorphism of E')

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$\rightsquigarrow$ Decompose large-degree isogenies into prime steps.
That is: Walk in an isogeny graph.

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In SageMath:

```
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sage: E
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Isogeny of degree 7
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sage: phi.rational_maps()
((x^7 + 129*x^6 - ... + 25)/(x^6 + 129*x^5 - ... + 36),
 (x^9*y - 16*x^8*y - ... + 70*y)/(x^9 - 16*x^8 + ...))
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Isogeny graphs

Consider a field k and let $S \not\ni \mathrm{char}(k)$ be a set of primes.

The *S*-isogeny graph over k consists of

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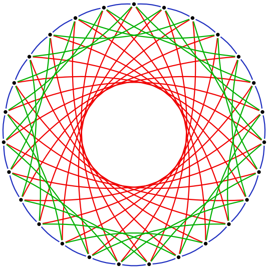
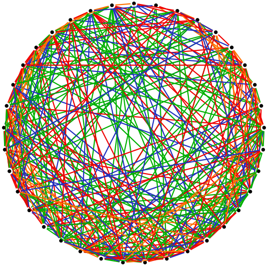
Example components containing $E\colon y^2 = x^3 + x$:



$k = \mathbb{F}_{419}, \; S = \{3, 5, 7\}$



$k = \mathbb{F}_{431^2}, \; S = \{2, 3, 5, 7\}$.

Predictable groups

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Let $p \geq 5$ be prime.

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- In that case, $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$ and $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}/(p+1) \times \mathbb{Z}/(p+1)$.

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$\rightsquigarrow$ Easy method to control the group structure by choosing p!

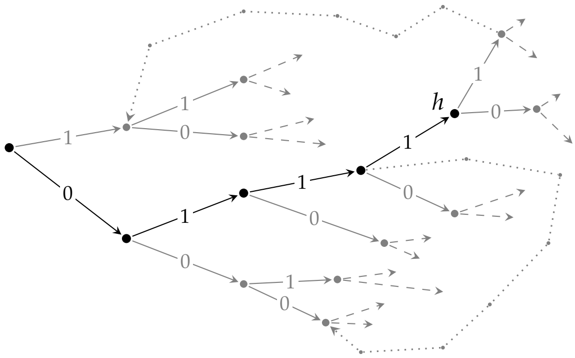
$\rightsquigarrow$ Cryptography works well using supersingular curves.

(All curves are supersingular until lunch time.)

Plan for this lecture

- High-level overview for intuition. ✓
- Elliptic curves & isogenies. ✓
- The CGL hash function.
- The CSIDH non-interactive key exchange.
- Hardness of isogeny problems, and reductions.
- The SQIsign signature scheme.
- Transcending to higher dimensions.

The Charles–Goren–Lauter hash function



- Start at some curve E.
- For each input digit b: Map the pair (E, b) to a finite subgroup $H \leq E$, compute $\varphi_H\colon E \to E'$, and set $E \leftarrow E'$.
- Finally return E.

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CSIDH [ˈsiːˌsaɪd]
[Castryck–Lange–Martindale–Panny–Renes 2018]

Isogeny-based key exchange: High-level view

E

Isogeny-based key exchange: High-level view

$$\begin{tikzcd}
E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E_A \\
E_B &
\end{tikzcd}$$

- Alice & Bob pick secret $\varphi_A \colon E \to E_A$ and $\varphi_B \colon E \to E_B$.
  (These isogenies correspond to walking on the isogeny graph.)

Isogeny-based key exchange: High-level view

$$\begin{tikzcd}
E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E_A \arrow[dl, dashed, shift left] \\
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- Alice and Bob transmit the end curves E_A and E_B.

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$$\begin{tikzcd}
E \arrow[rr, "\varphi_A"] \arrow[dd, "\varphi_B"'] & & E_A \arrow[d, "\varphi_B'"] \arrow[ddll, dashed, bend left=5] \\
 & & E_{AB} \\
E_B \arrow[r, "\varphi_A'"'] \arrow[uurr, dashed, bend left=5] & E_{BA} &
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- Alice <u>somehow</u> finds a "parallel" $\varphi_{A'} : E_B \to E_{BA}$, and Bob <u>somehow</u> finds $\varphi_{B'} : E_A \to E_{AB}$,

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- Alice <u>somehow</u> finds a "parallel" $\varphi_{A'} : E_B \to E_{BA}$, and
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How to find "parallel" isogenies?

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CSIDH’s solution:

Use special isogenies φ_A which can be transported to the curve E_B totally independently of the secret isogeny φ_B.

(Similarly with reversed roles, of course.)

“Special” isogenies

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We consider prime ℓ and refer to φ_ℓ as a “special” isogeny.

Cycles from "special" isogenies

What happens when we iterate such a "special" isogeny?

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$$\begin{tikzcd}
 & & & E_{\ell^{r-1}} \arrow[d] & \cdots \arrow[l] & E_{\ell^6} \arrow[l] \\
 & & & E_{\ell^r} \arrow[d] & & E_{\ell^5} \arrow[u] \\
E \arrow[r] & E_{\ell} \arrow[r] & E_{\ell^2} \arrow[r] & E_{\ell^3} \arrow[r] & E_{\ell^4} \arrow[ur] &
\end{tikzcd}$$

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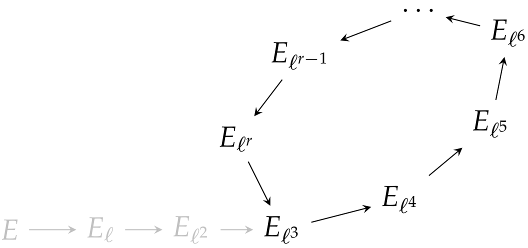

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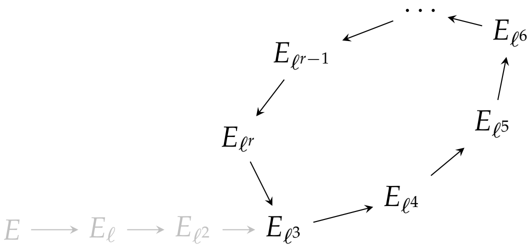

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- Exercise: Each curve has only one other rational ℓ-isogeny.

!! Reverse arrows are unique; the "tail" $E \to E_{\ell^3}$ cannot exist.

$\implies$ The "special" isogenies φ_ℓ form isogeny cycles!

Compatible cycles from "special" isogenies

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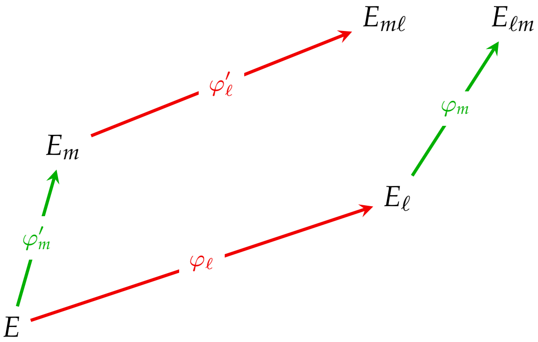

Compatible cycles from "special" isogenies

What happens when we compose those "special" isogenies?

$$\begin{tikzcd}
 & E_{m\ell} & E_{\ell m} \\
E_m \arrow[ur, "\varphi'_\ell"] & E_\ell \arrow[ur, "\varphi_m"] & \\
E \arrow[u, "\varphi'_m"] \arrow[ur, "\varphi_\ell"'] & &
\end{tikzcd}$$

- Exercise: $\ker(\varphi'_\ell \circ \varphi'_m) = \ker(\varphi_m \circ \varphi_\ell) = \langle \ker \varphi_\ell, \ker \varphi'_m \rangle$.

Compatible cycles from "special" isogenies

What happens when we compose those "special" isogenies?

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E_m \arrow[r, "\varphi'_\ell"] & E_{\ell m} \\
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- Exercise: $\ker(\varphi'_\ell \circ \varphi'_m) = \ker(\varphi_m \circ \varphi_\ell) = \langle \ker \varphi_\ell, \ker \varphi'_m \rangle$.

!! The order cannot matter $\implies$ cycles must be compatible.

CSIDH in one slide

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- Choose some small odd primes $\ell_1, ..., \ell_n$.
- Make sure $p = 4 \cdot \ell_1 \cdots \ell_n - 1$ is prime.

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- Look at the "special" ℓ_i-isogenies within X.

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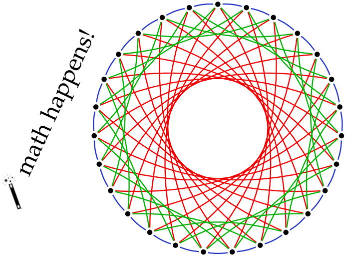

$p = 419$
$\ell_1 = 3$
$\ell_2 = 5$
$\ell_3 = 7$

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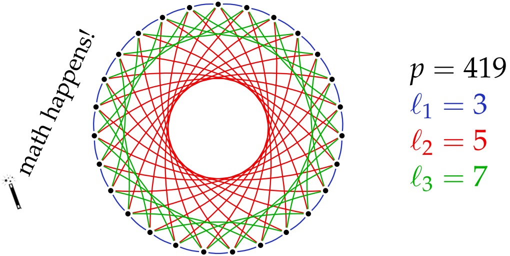
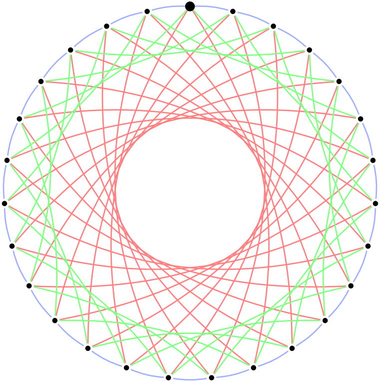


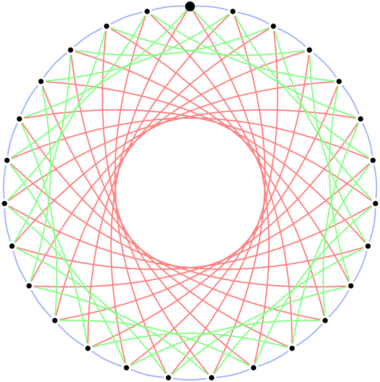
- Walking "left" and "right" on any ℓ_i-subgraph is efficient.

CSIDH key exchange

Alice
[**+**, **+**, **−**, **−**]



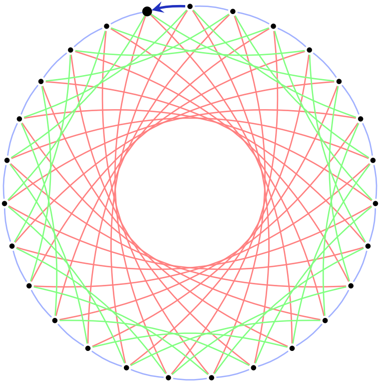
Bob
[**−**, **+**, **−**, **−**]



CSIDH key exchange

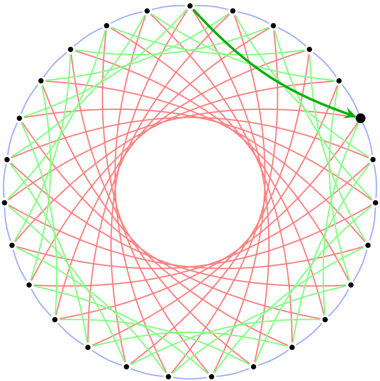
Alice

$[\mathbf{+}, \mathbf{+}, \mathbf{-}, \mathbf{-}]$
$\uparrow$



Bob

$[\mathbf{-}, \mathbf{+}, \mathbf{-}, \mathbf{-}]$
$\uparrow$



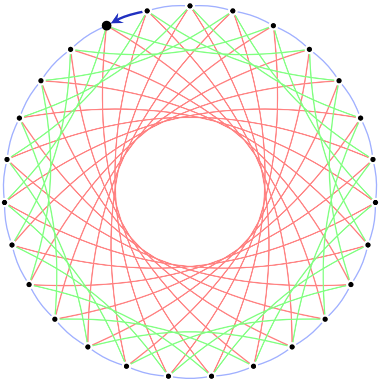
CSIDH key exchange

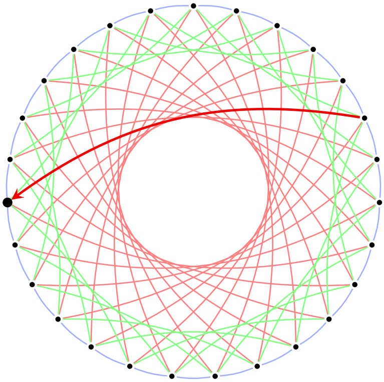
Alice

$[+, +, -, -]$
$\quad\ \uparrow$

Bob

$[-, +, -, -]$
$\quad\ \uparrow$

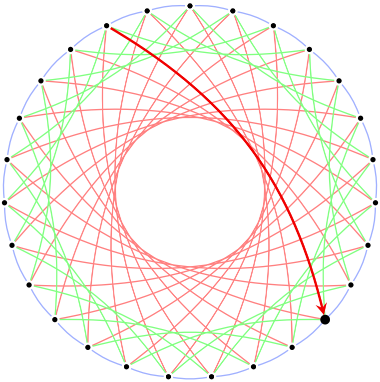




CSIDH key exchange

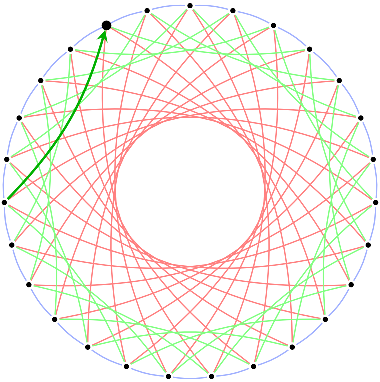
Alice

[+, +, −, −]



Bob

[−, +, −, −]



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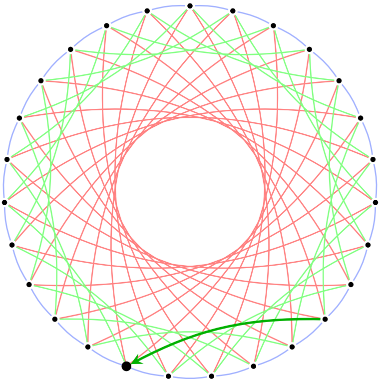

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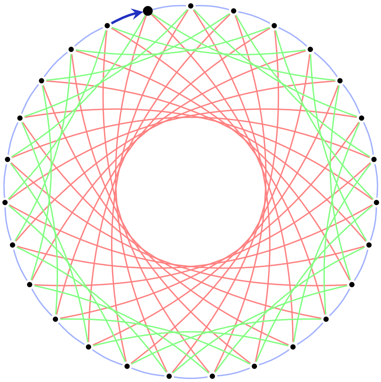

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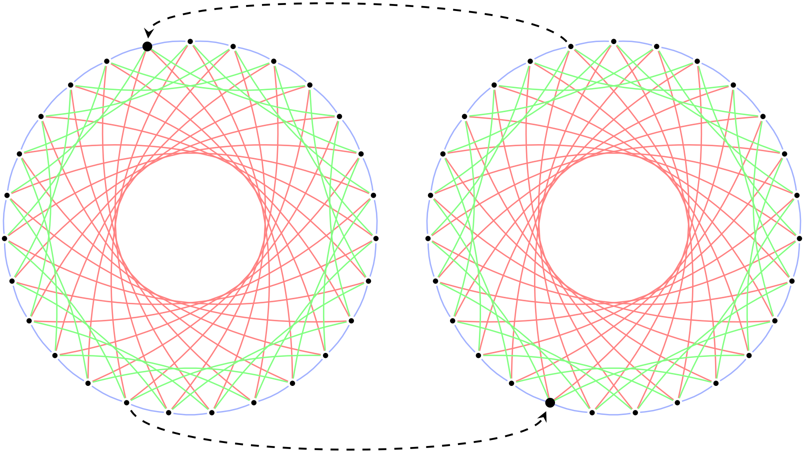

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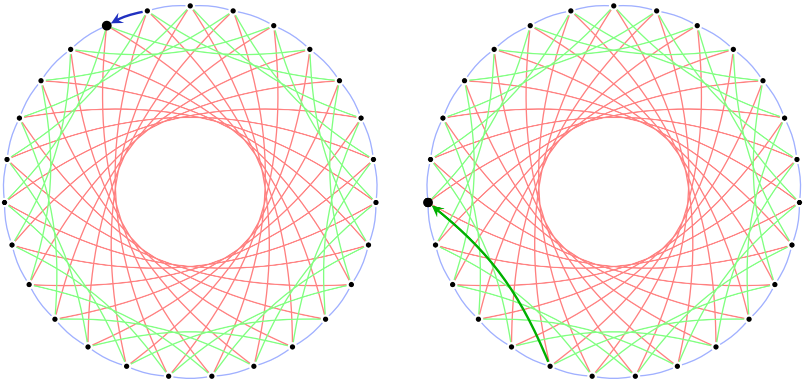

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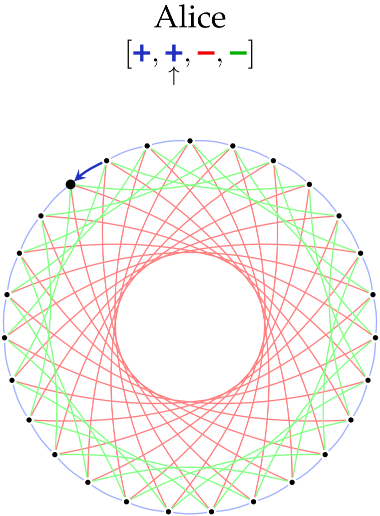

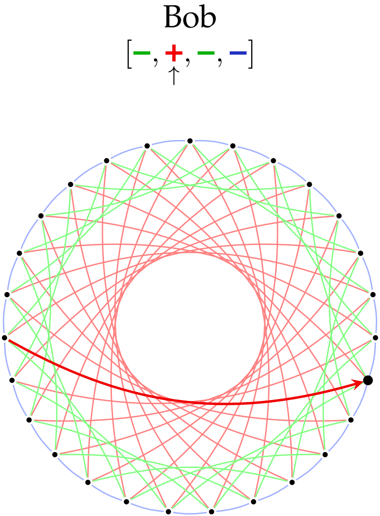

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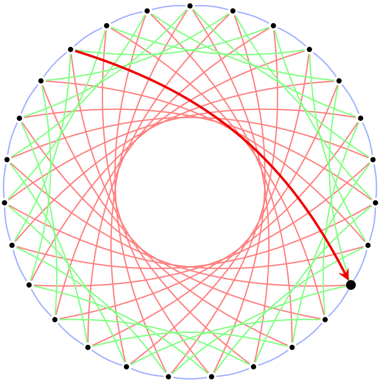

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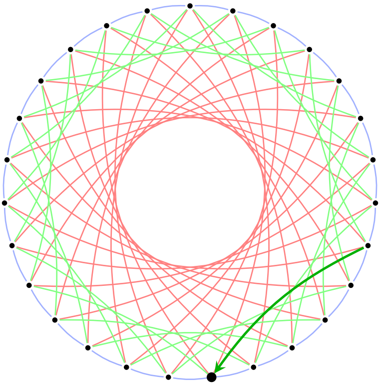

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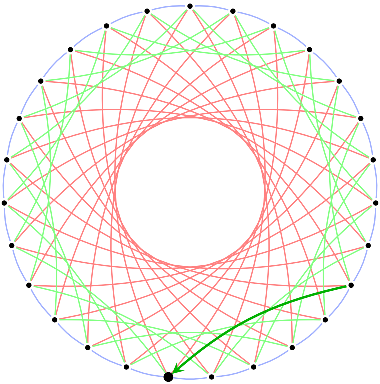

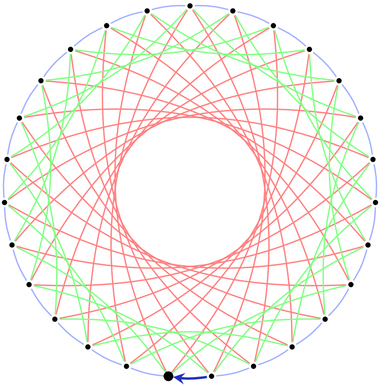

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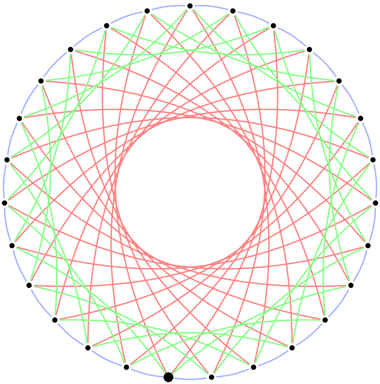

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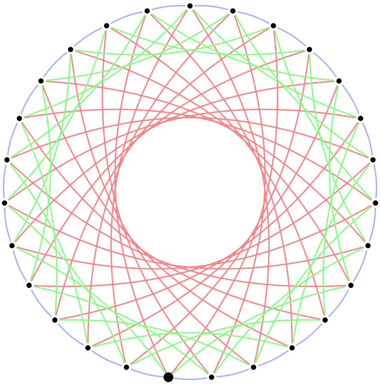

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Example: $[+, +, -, -, -, +, -, -]$ just becomes $(+1, \;\; 0, -3) \in \mathbb{Z}^3$.

There is a group action of $(\mathbb{Z}^n, +)$ on our set of curves X!

(An action of a group $(G, \cdot)$ on a set X is a map $*\colon G \times X \to X$
such that $id * x = x$ and $g * (h * x) = (g \cdot h) * x$ for all $g, h \in G$ and $x \in X$.)

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!! This group characterizes *when two paths lead to the same curve*.

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(Finding a point of order ℓ_i: Pick $x \in \mathbb{F}_p$ random. Find $y \in \mathbb{F}_{p^2}$ such that $P = (x, y) \in E$. Compute $Q = [\frac{p+1}{\ell_i}]P$. Hope that $Q \neq \infty$, else retry.)

In SageMath:

```
sage: E = EllipticCurve(GF(419^2), [1,0])
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Elliptic Curve defined by y^2 = x^3 + x
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sage: P
(218 : 403 : 1)
sage: P.order().factor()
2 * 3 * 7
sage: EE = E.isogeny_codomain(2*3*P)  # "left" 7-step
sage: EE
Elliptic Curve defined by y^2 = x^3 + 285*x + 87
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Efficient x-only arithmetic

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The same reasoning works for isogeny formulas.

Net result: With x-only arithmetic everything happens over $\mathbb{F}_p$.
$\Longrightarrow$ (Relatively) efficient CSIDH implementations!

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For group <u>actions</u>, we simply cannot compose $a * s$ and $b * s$!

Plan for this lecture

- High-level overview for intuition. ✓
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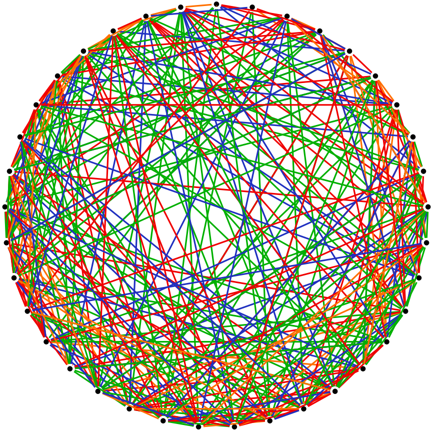

Now:

Supersingular isogeny graphs over $\mathbb{F}_{p^2}$.

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<u>Bottom line</u>: Fully exponential. Complexity $\exp\big((\log p)^{1+o(1)}\big)$.

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Theorem (Wesolowski 2021): Assuming GRH, the isogeny and endomorphism-ring problems are *polynomial-time equivalent*.

SoK: Isogeny problems

Some isogeny problems are much more broken than others.

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Is SIKE broken yet?

Home About

Schemes

Name	Type	Classical Security	Quantum Security	References	Additional Information
▹ SIDH	Key Exchange	$\tilde{O}(n^3)$	$\tilde{O}(n^3)$	JDF11 DJP14 CLN16	▹ Comment
SIKE	KEM	$\tilde{O}(n^3)$	$\tilde{O}(n^3)$	SIKE	▹ Comment
B-SIDH	Key Exchange	$\tilde{O}(n^3)$	$\tilde{O}(n^3)$	Cos19	▹ Comment
CRS	Key Exchange, Non Interactive Key Exchange	$\exp(n)^{1/2}$	L(1/2)	Cou06 RS06 DKS18	▹ Comment
CSIDH	Key Exchange, Non Interactive Key Exchange	$\exp(n)^{1/2}$	L(1/2)	CL+18 CD19	▹ Comment

https://issikebrokenyet.github.io

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:) We understand how I_φ, I_ψ relate for isogenies $\varphi, \psi\colon E \to E'$.
(NB: Same E'.)

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$\ddot\smile$ One direction is easy, the other seems hard! $\rightsquigarrow$ *Cryptography!*

The Deuring correspondence (examples)

Let $p = 7799999$ and let $\mathbf{i}, \mathbf{j}$ satisfy $\mathbf{i}^2 = -1,\ \mathbf{j}^2 = -p,\ \mathbf{ji} = -\mathbf{ij}$.

The ring $\mathcal{O}_0 = \mathbb{Z} \oplus \mathbb{Z}\,\mathbf{i} \oplus \mathbb{Z}\,\frac{\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{1+\mathbf{ij}}{2}$
corresponds to the curve $E_0\colon\ y^2 = x^3 + x$.

The ring $\mathcal{O}_1 = \mathbb{Z} \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+32631010\mathbf{i}+\mathbf{ij}}{9894}$
corresponds to the curve $E_1\colon\ y^2 = x^3 + 1$.

The ideal $I = \mathbb{Z}\,4947 \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{598+4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+598\mathbf{i}+\mathbf{ij}}{2}$
defines an isogeny $E_0 \to E_1$ of degree $4947 = 3 \cdot 17 \cdot 97$.

Signing with isogenies

- Fiat–Shamir: signature scheme from identification scheme.

$$E_0 \xrightarrow{\text{secret}} E_A$$

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$$\begin{tikzcd}
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- Easy signature: $E_A \to E_0 \to E_1 \to E_2$. *Obviously broken.*
- <u>**SQIsign**'s solution</u>: Construct new path $E_A \to E_2$ (using *secret*).

SQIsign

Main idea:

- Construct the "signature square" in quaternion land.
- Project the secret and signature down to the curve world.
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> *"If you have KLPT implemented very nicely as a black box, then anyone can implement SQIsign."*
> — Yan Bo Ti

SQIsign: Numbers

sizes

parameter set	public keys	signatures
NIST-**I**	**64** bytes	**177** bytes
NIST-**III**	**96** bytes	**263** bytes
NIST-**V**	**128** bytes	**335** bytes

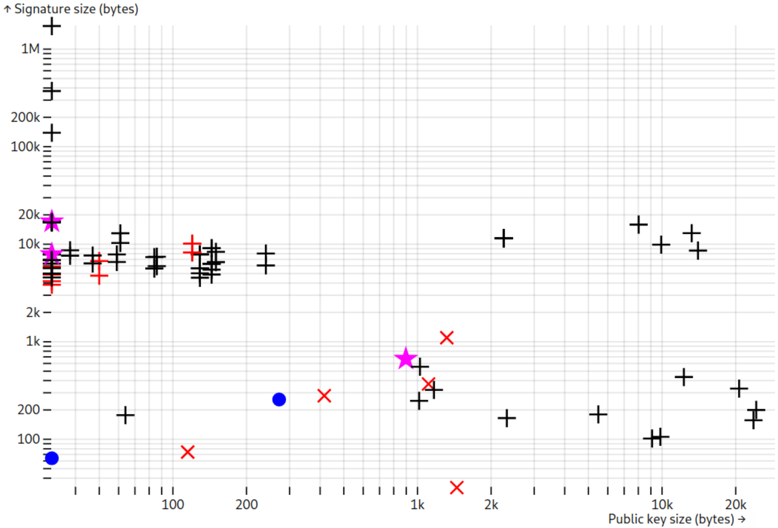
performance

Cycle counts for a *generic C implementation* running on an Intel *Ice Lake* CPU.
Optimizations are certainly possible and work in progress.

parameter set	keygen	signing	verifying
NIST-**I**	**3728** megacycles	**5779** megacycles	**108** megacycles
NIST-**III**	**23734** megacycles	**43760** megacycles	**654** megacycles
NIST-**V**	**91049** megacycles	**158544** megacycles	**2177** megacycles

Source: `https://sqisign.org`

SQIsign: Comparison



Source: `https://pqshield.github.io/nist-sigs-zoo`

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Main task in **SQIsign verification**:

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- Graph view: Each ψ_i is a step in the ℓ-isogeny graph.

In SageMath:

```
sage: E = EllipticCurve(GF(2^127-1), [1,0])
sage: K = E(23, 40490046516039691075571867486180936666)
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Composite morphism of degree 1063...6608 = 2^123:
  From: Elliptic Curve defined by y^2 = x^3 + x
                over Finite Field of size 1701...5727
  To:   Elliptic Curve defined by
                y^2 = x^3 + 1625...8575*x + 1200...7360
                over Finite Field of size 1701...5727
```

Strategies for composite-degree isogenies

Recall: We split ℓ^n-isogenies into n individual ℓ-isogenies ψ_i.
This requires computing $K_i := [\ell^{n-i}](\psi_{i-1} \circ \cdots \circ \psi_1)(n)$ for all i.

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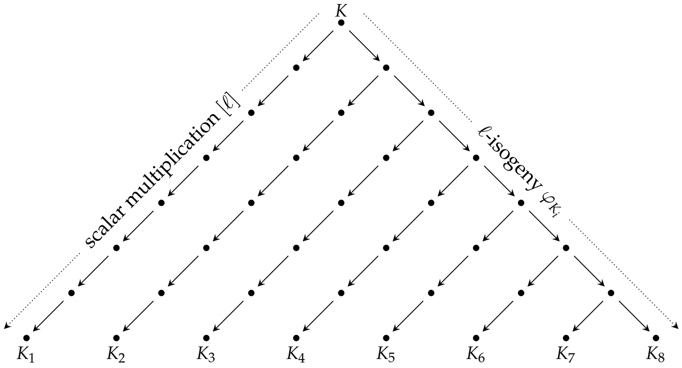
Naïve strategy:



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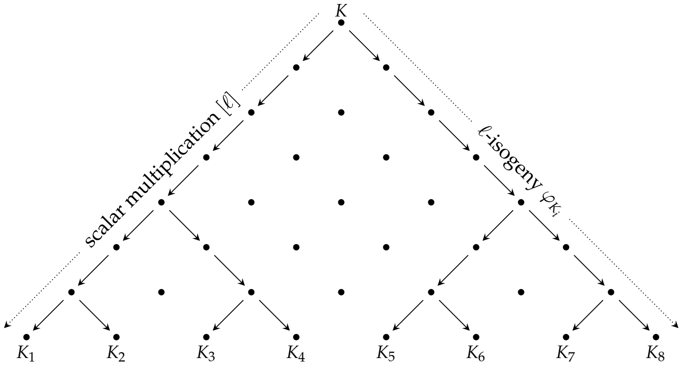
Sparse strategy:



Optimal strategies for composite-degree isogenies

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Similar techniques exist for general *composite* degree.

Plan for this lecture

- High-level overview for intuition. ✓
- Elliptic curves & isogenies. ✓
- The CGL hash function. ✓
- The CSIDH non-interactive key exchange. ✓
- Hardness of isogeny problems, and reductions. ✓
- The SQIsign signature scheme. ✓
- Transcending to higher dimensions.

Gluing elliptic curves

- Fallout from the SIDH attack: New tools.

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Main technique underlying attack:

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- The product $E \times E'$ is an abelian *surface*.
  Compare: A product of two lines is a plane!
- Similar to elliptic curves in many ways:
  - Points form an abelian group.
  - Similar group structure, but more components.
  - Can define isogenies from kernel subgroups.

The embedding lemma

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2.1. **The embedding lemma.** If α_1, α_2 are two endomorphisms of an elliptic curve E of degree a_1 and a_2, then $\alpha_1 \circ \alpha_2$ is of degree $a_1 a_2$. However it is harder to control the degree of the sum; by Cauchy-Schwartz we can bound it as: $(a_1^{1/2} - a_2^{1/2})^2 \leq \deg(\alpha_1 + \alpha_2) \leq (a_1^{1/2} + a_2^{1/2})^2$ (unless $\alpha_1 = -\alpha_2$). And $\alpha_1 + \alpha_2$ is of degree $a_1 + a_2$ if and only if $\alpha_1 \tilde{\alpha}_2$ is of trace 0.

If α_1 commutes with α_2, we can instead use Kani's lemma [Kan97, § 2] to build an endomorphism F *in dimension* 2 on E^2 which is an $(a_1 + a_2)$-isogeny (so is of degree $(a_1 + a_2)^2$ since we are in dimension 2). So by going to higher dimension we can combine degrees additively. The proof of this lemma is very simple (a simple two by two matrix computation), but its powerful algorithmic potential went unnoticed until Castrick and Decru applied it in [CD22] to attack on SIDH.

—Damien Robert [ePrint 2022/1704]

The embedding lemma

Consider a commutative diagram of isogenies

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\psi"'] & E' \arrow[d, "\psi'"] \\ E'' \arrow[r, "\varphi'"'] & E''' \end{tikzcd}$$

where $a := \deg \varphi$ and $b := \deg \psi$ are coprime; let $N := a + b$.

Lemma. Then

$$F := \begin{pmatrix} \varphi & \widehat{\psi'} \\ -\psi & \widehat{\varphi'} \end{pmatrix}$$

defines an N-isogeny $E \times E''' \to E' \to E''$.

Its kernel is $\ker(F) = \left\{ (\widehat{\varphi}(P), \psi'(P)) \mid P \in E'[N] \right\}$.

Representing $\varphi|_{E[N]}$

Recall: For embedding lemma, need to evaluate φ on $E[N]$.

$\rightsquigarrow$ Exponentially many points. ☹

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Clever trick:

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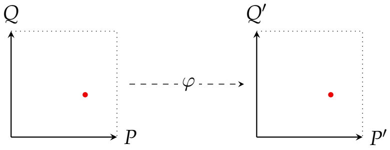

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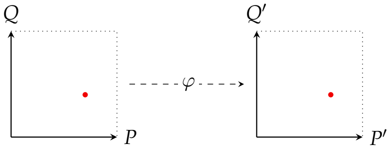

Evaluating φ at an arbitrary point $T \in E[N]$:

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   This is a DLP-like computation, which is easy whenever N is smooth!
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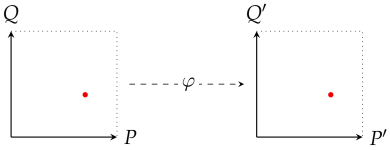

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$\Longrightarrow$ The data (P, Q, P', Q') encodes the *restriction* $\varphi|_{E[N]}$.

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Ad

https://isogeny.club

Questions?

(Also feel free to email me: `lorenz@yx7.cc`)