Introduction to isogeny-based cryptography

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Big picture 🔎🔎

- Isogenies are a type of maps between elliptic curves.

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- Sampling an isogeny *from* some curve is easy, recovering an isogeny *between* given curves seems very hard.

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$\rightsquigarrow$ ***Cryptography!***

[Modern isogeny-based cryptography uses not just elliptic curves, but also higher-dimensional abelian varieties. $\rightsquigarrow$ *Friday.* ☺]

Plan for this talk

- Some high-level intuition.
- Elliptic curves & isogenies.
- Isogeny group actions.
- Signatures from isogenies.

Diffie–Hellman key exchange (1976)

Public parameters:

- a finite group G (traditionally $\mathbb{F}_p^*$, today elliptic curves)
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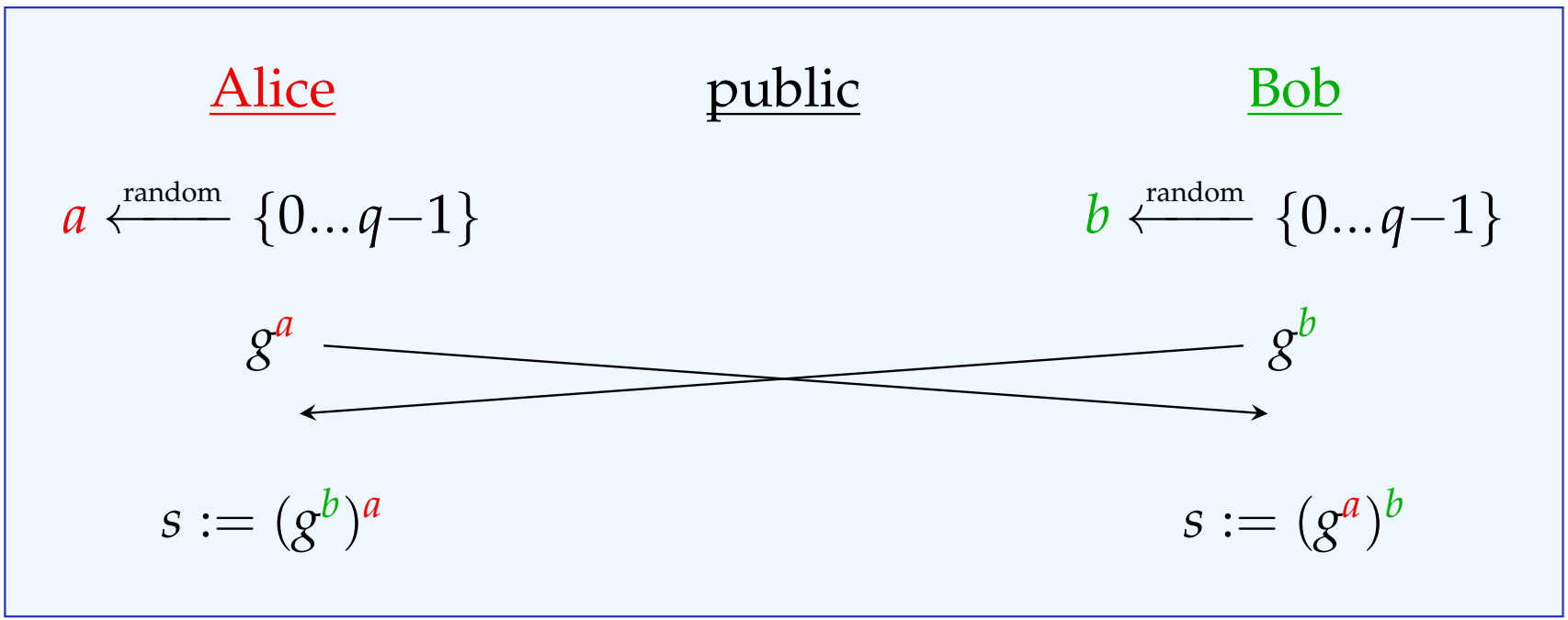

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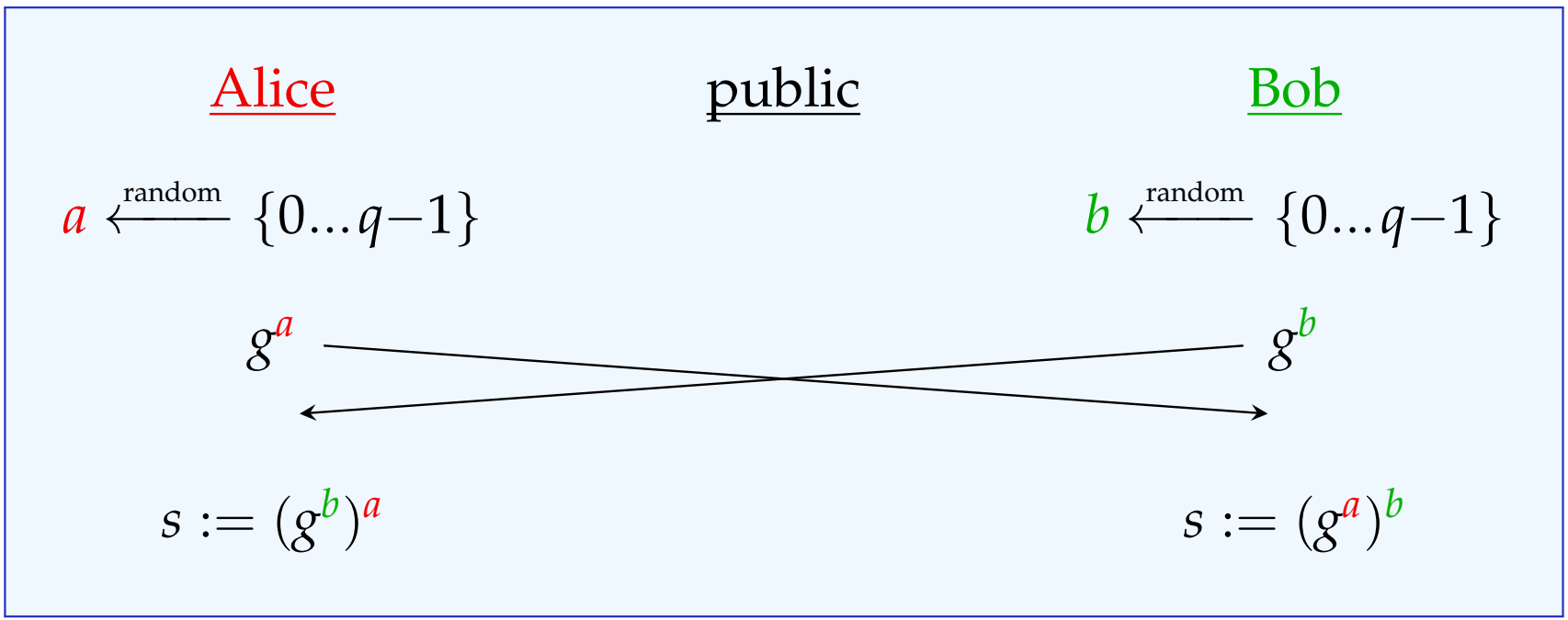

Fundamental reason this works: $\cdot^a$ and $\cdot^b$ are commutative!

Diffie–Hellman: Bob vs. Eve

Bob

1. Set $t \leftarrow g$.
2. Set $t \leftarrow t \cdot g$.
3. Set $t \leftarrow t \cdot g$.
4. Set $t \leftarrow t \cdot g$.

...

$b-2$. Set $t \leftarrow t \cdot g$.

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b. Publish $B \leftarrow t \cdot g$.

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Is this a good idea?

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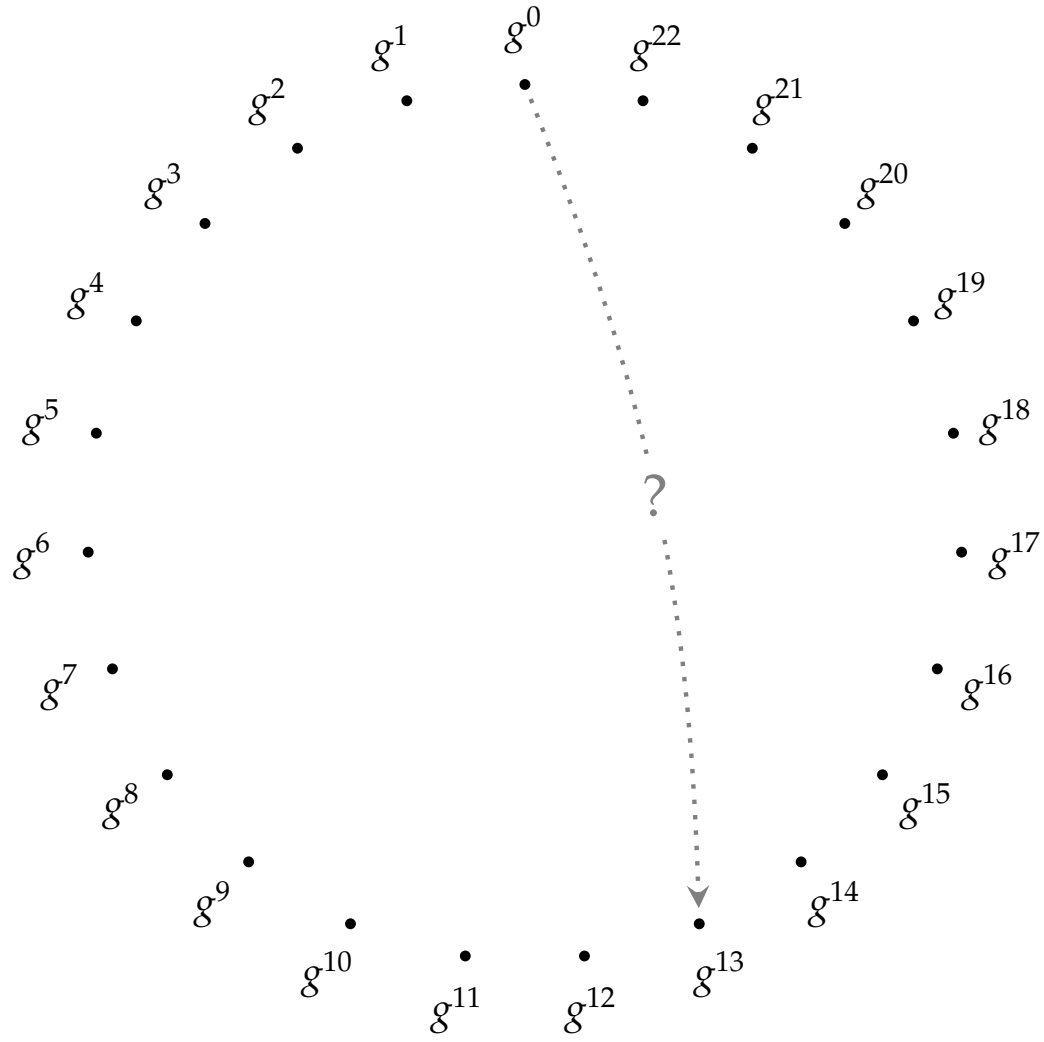
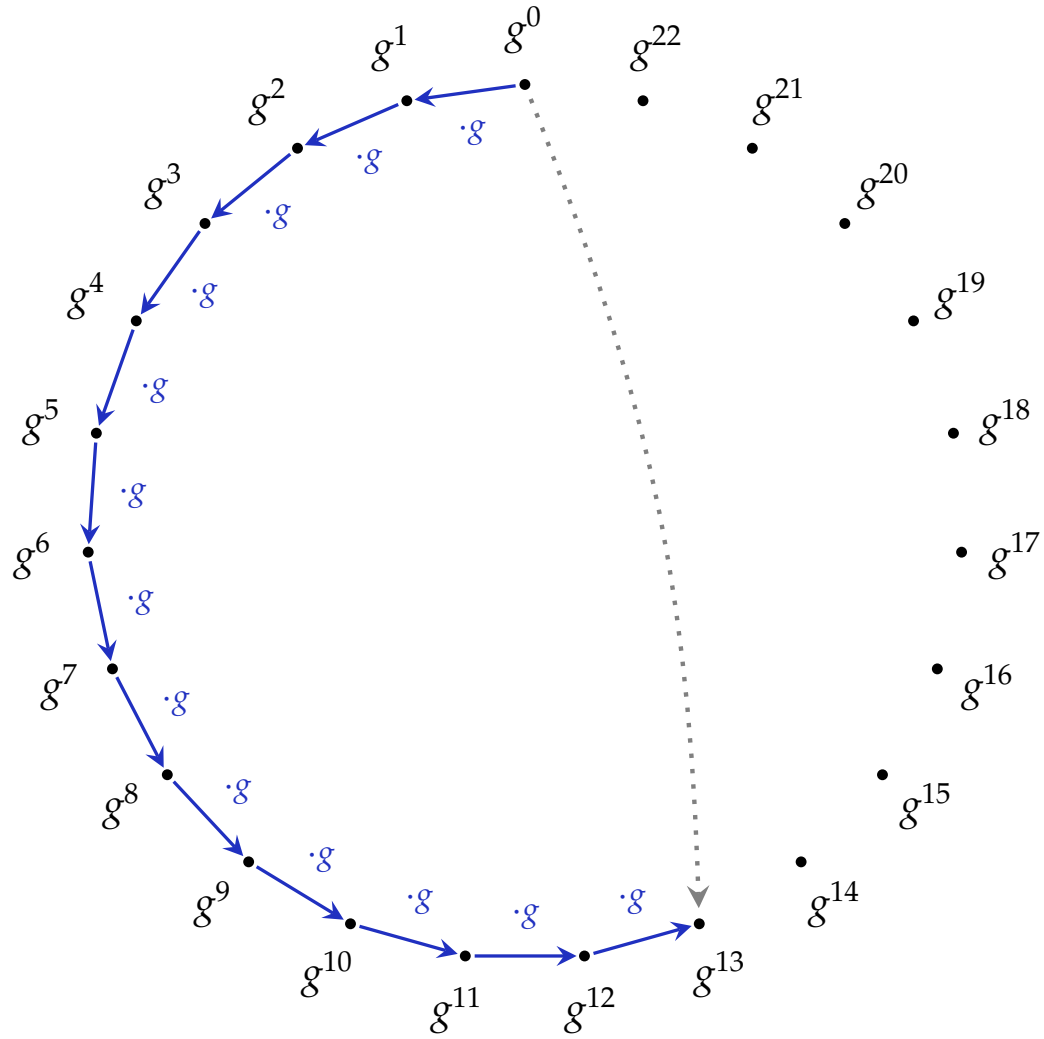
Effort for both: $O(\#G)$. Bob needs to be smarter.

(This attacker is also kind of dumb, but that doesn't matter for my point here.)



Bob computes his public key g^{13} from g.

multiply



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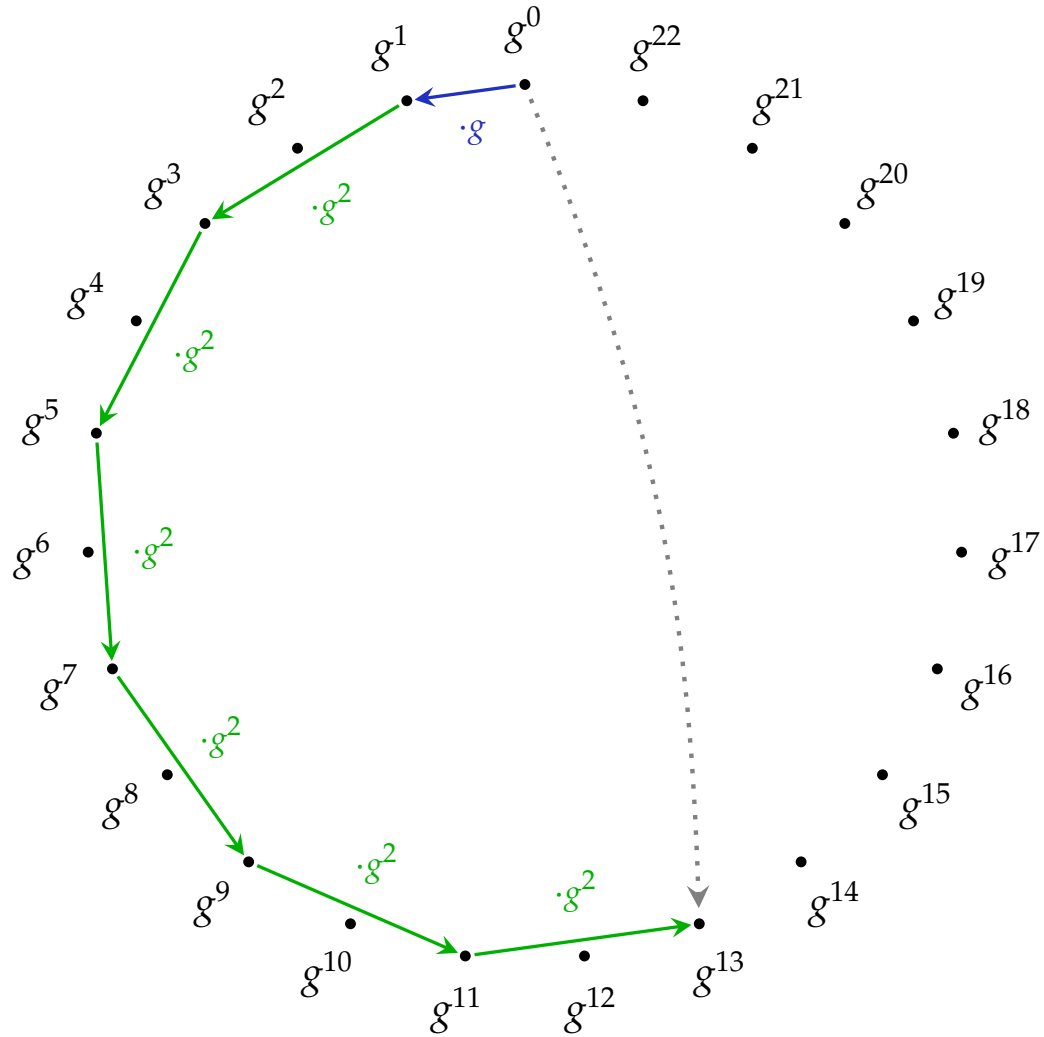
Square-and-multiply



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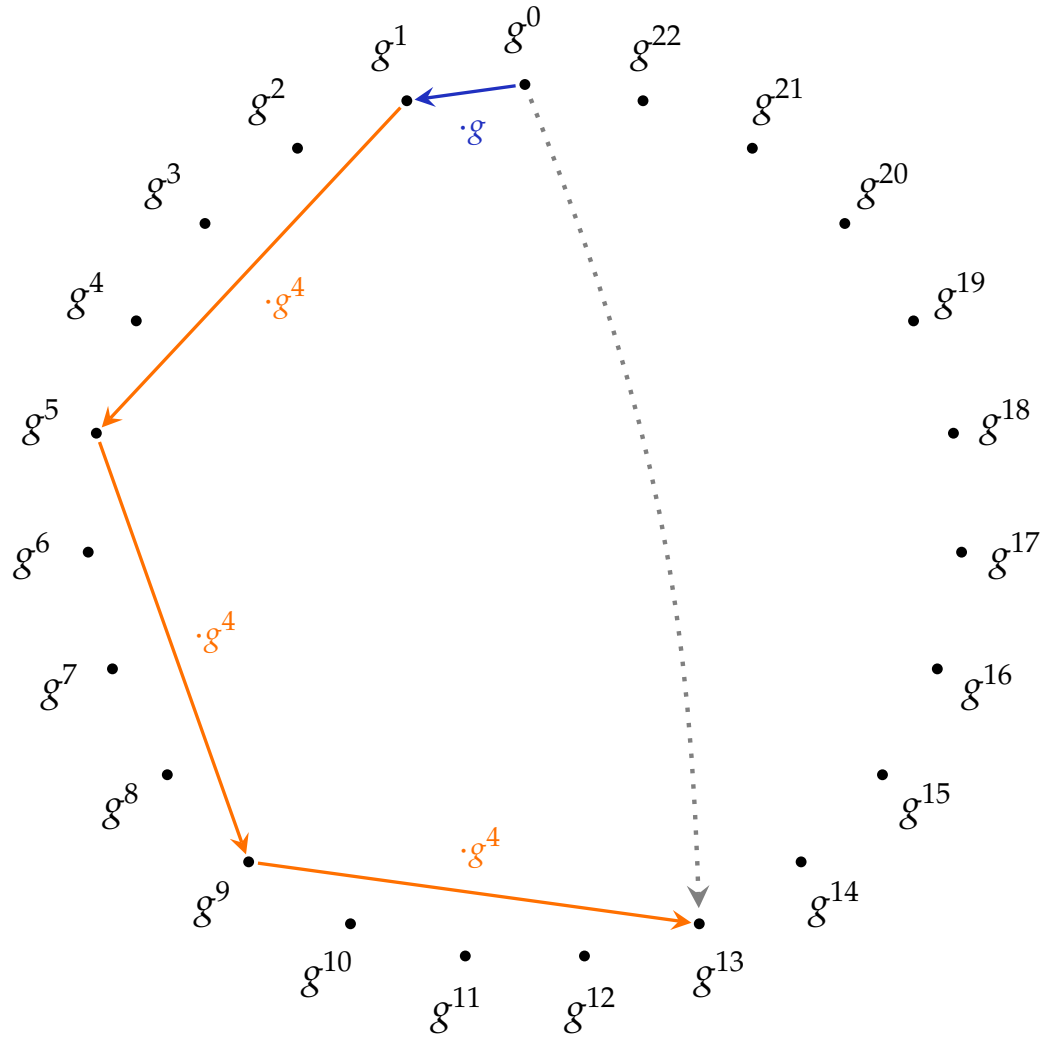
Square-and-multiply-and-square-and-multiply



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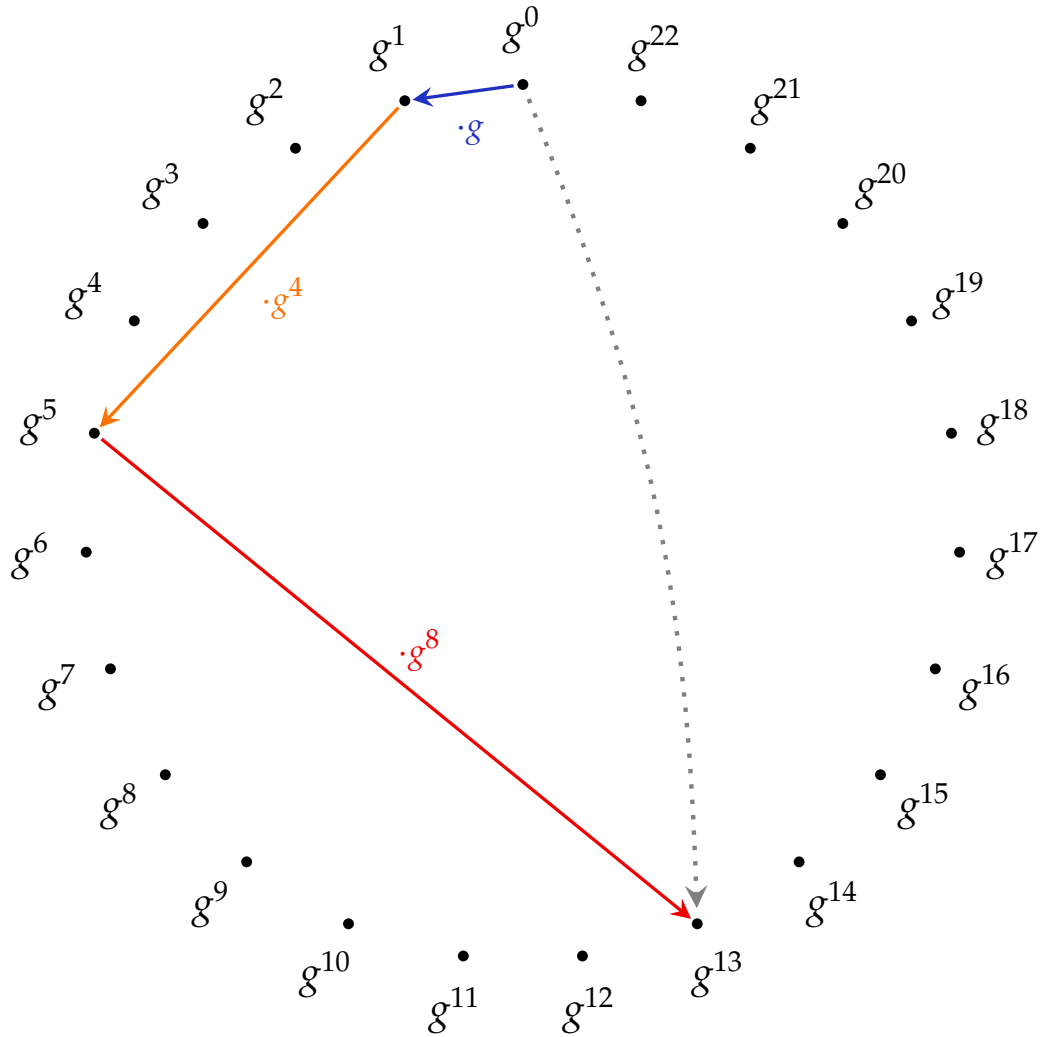

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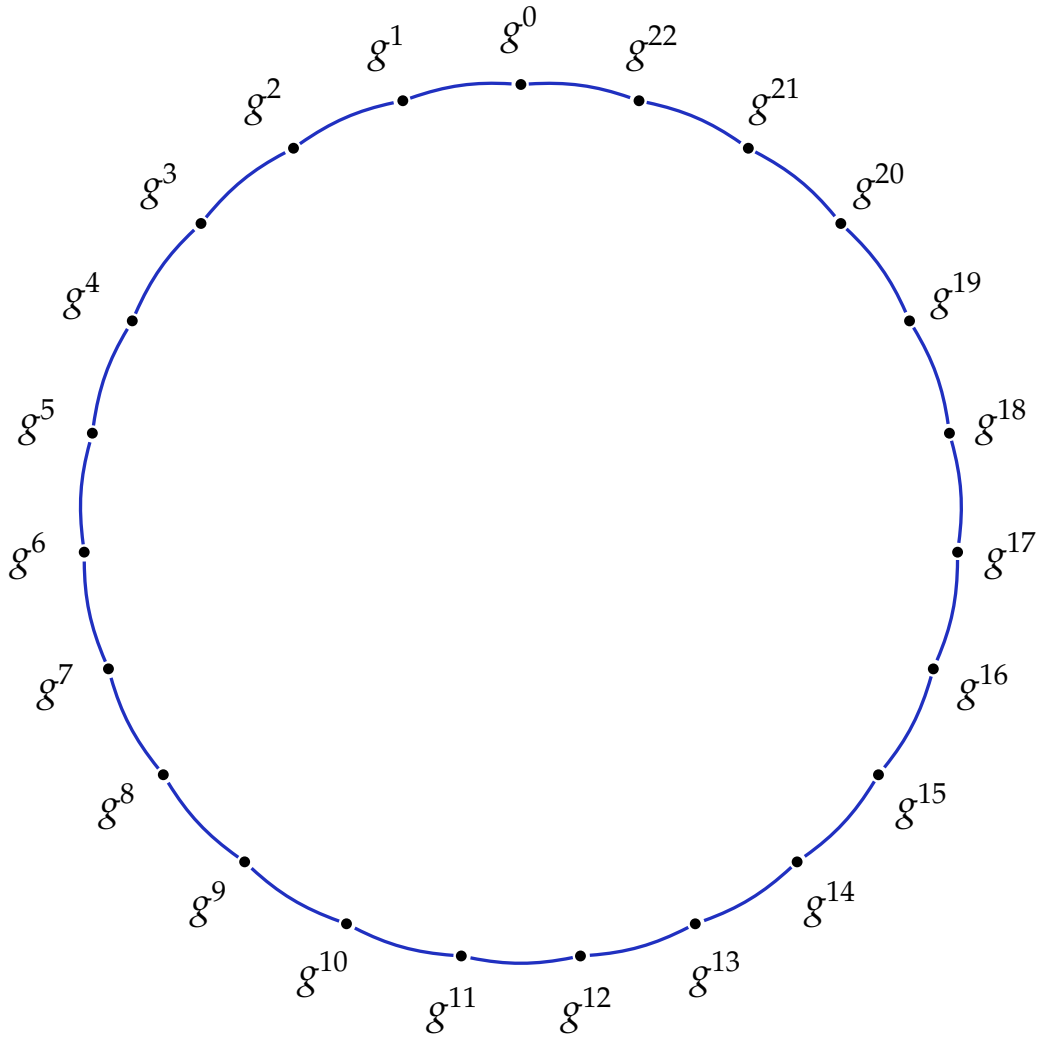
Square-and-multiply as graphs



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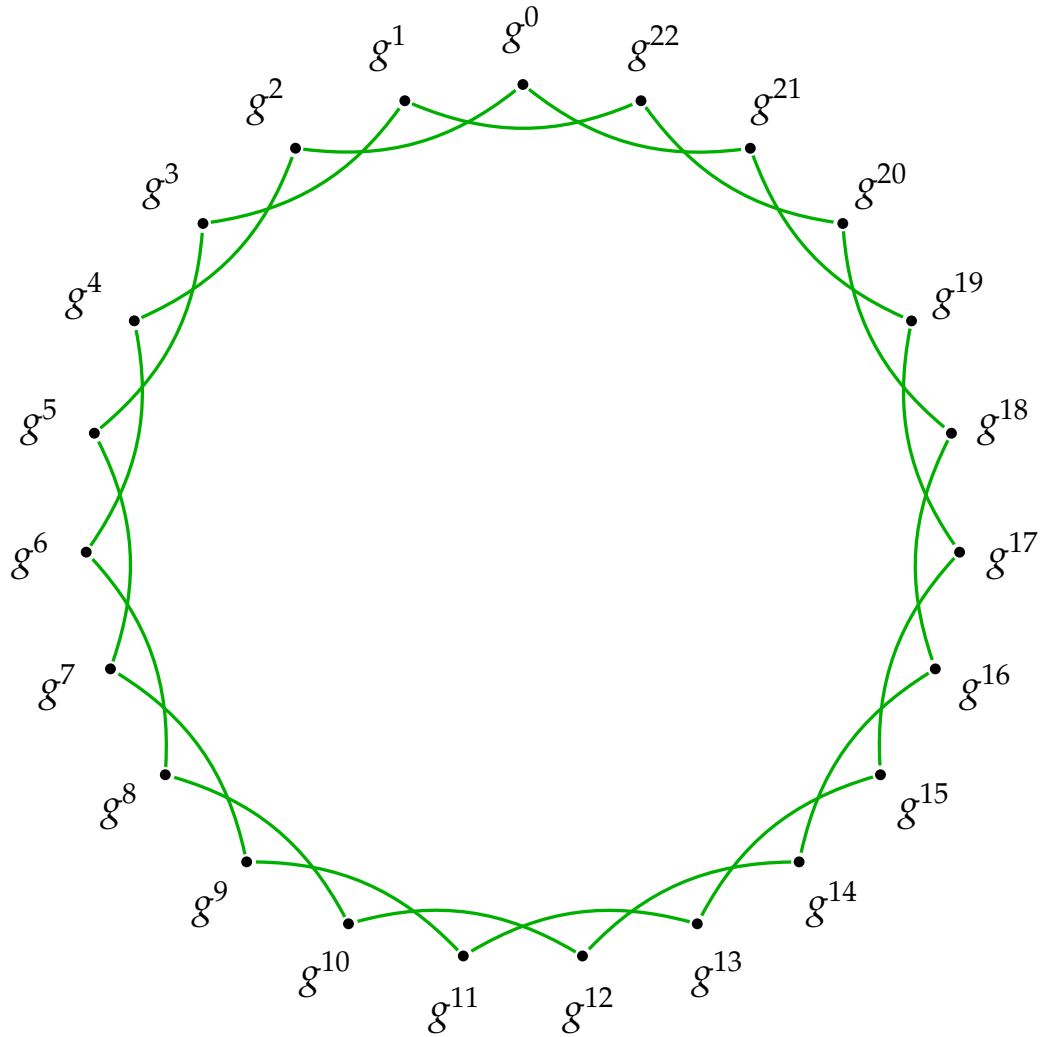

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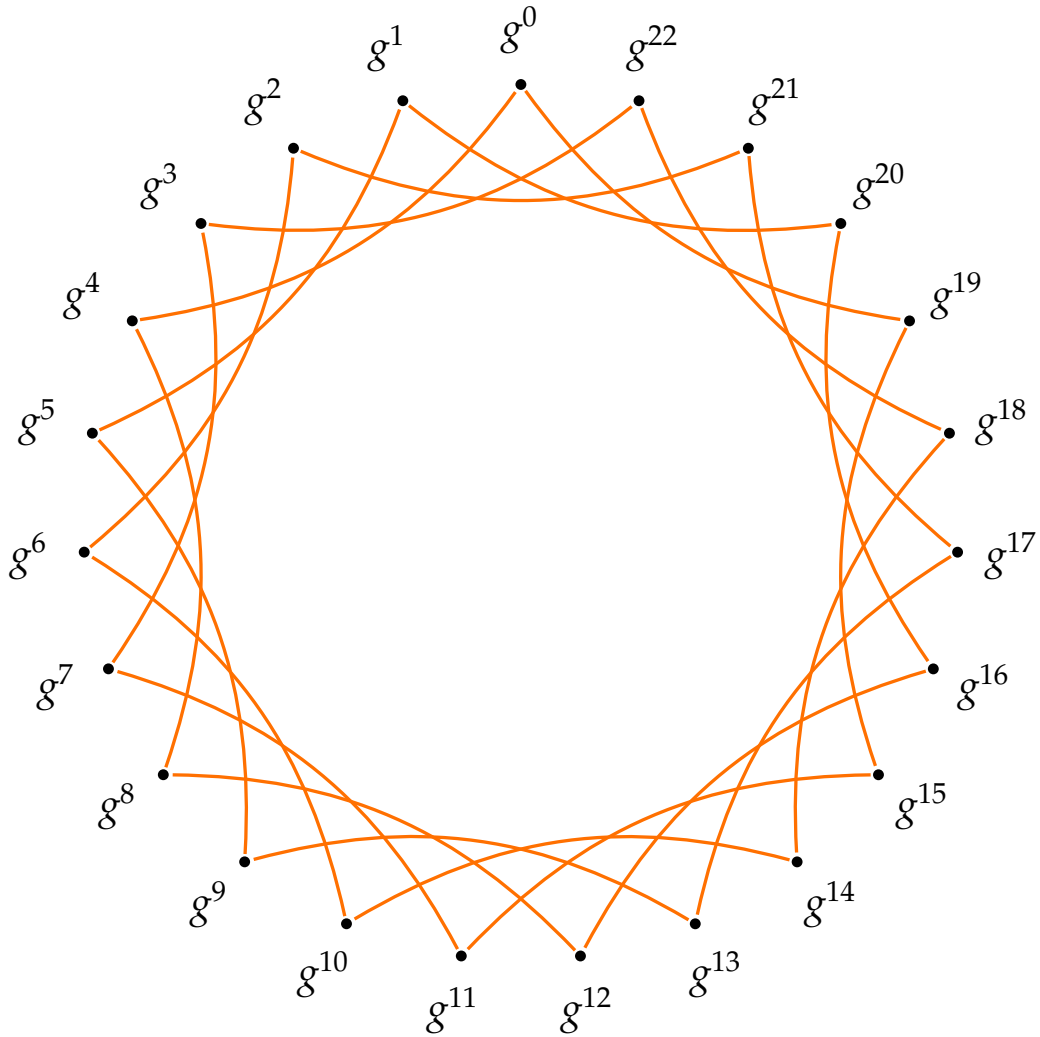

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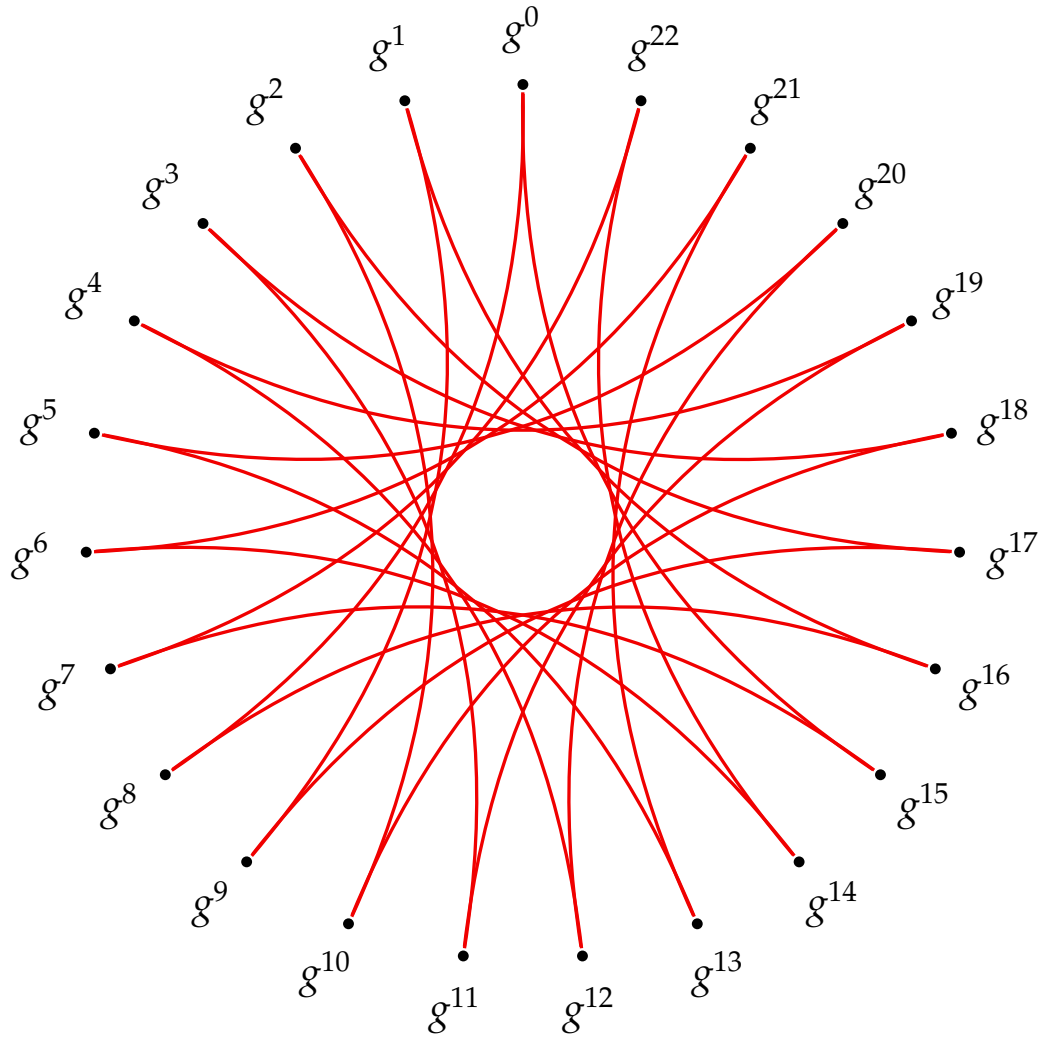

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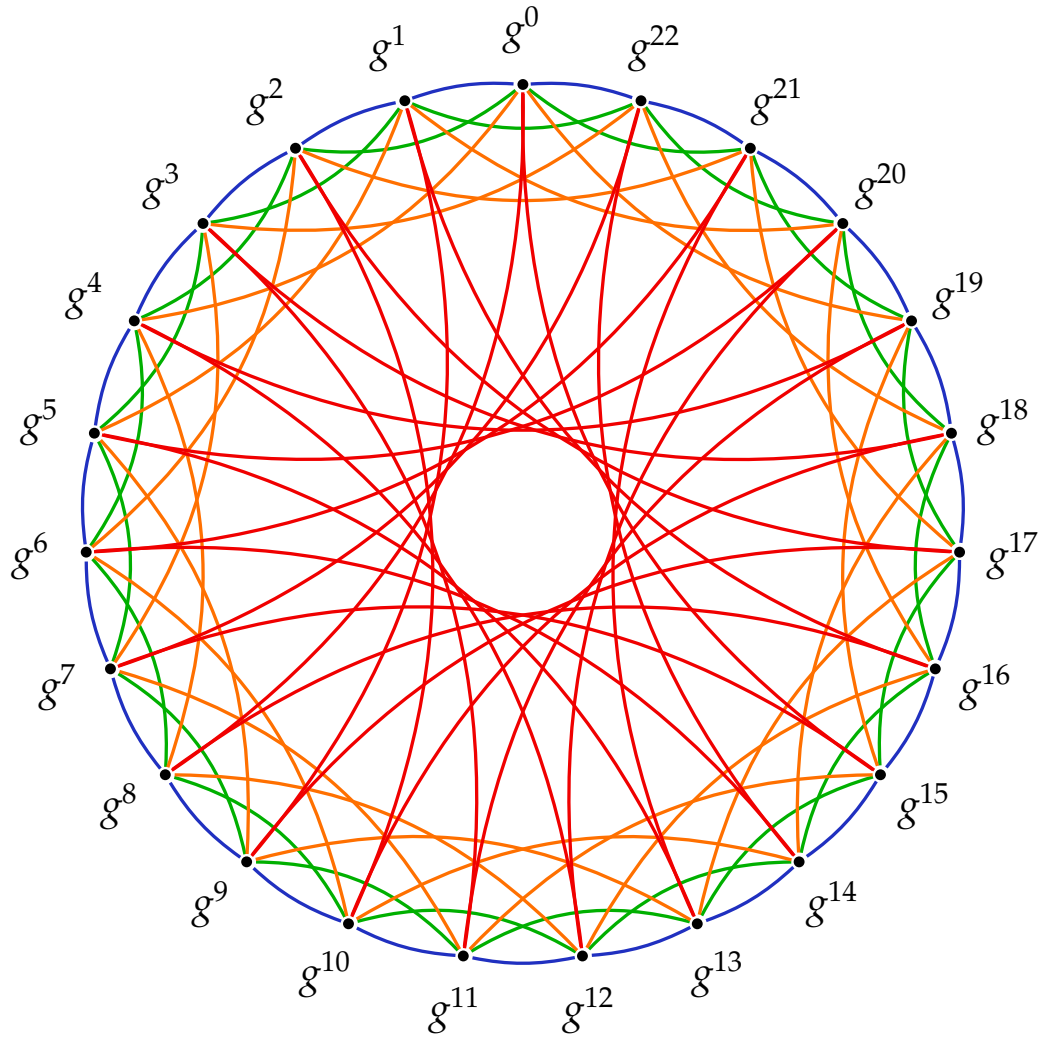

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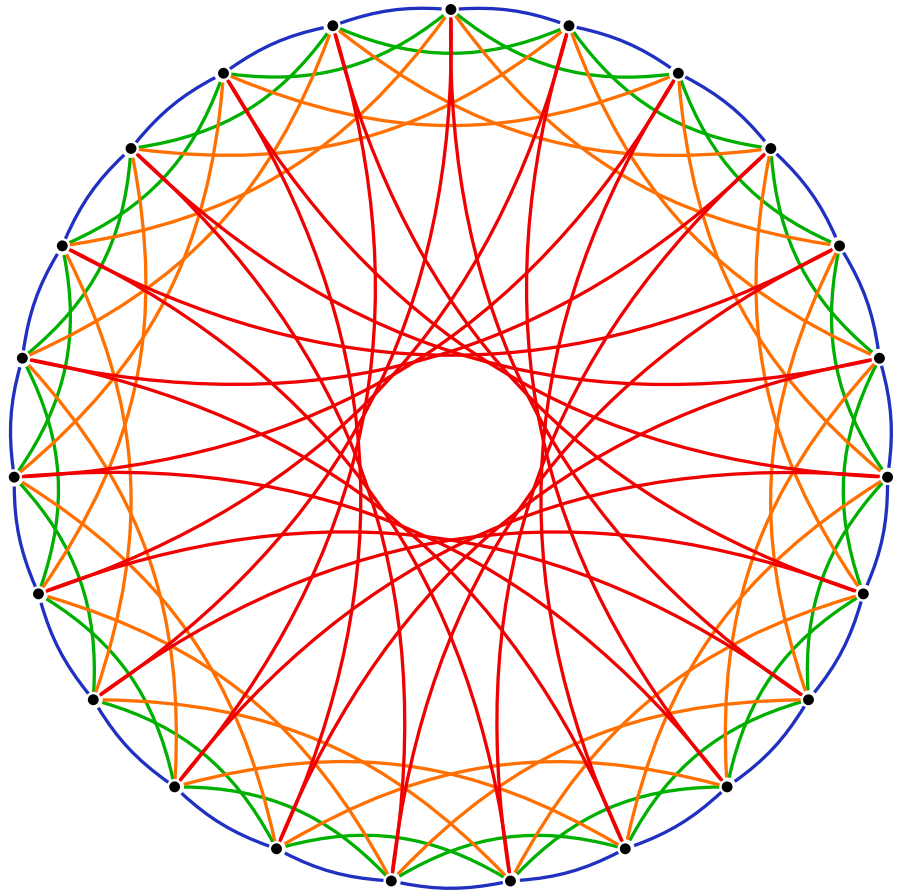

Fast mixing: paths of length log(# nodes) to everywhere.

Shor's algorithm vs. DLP

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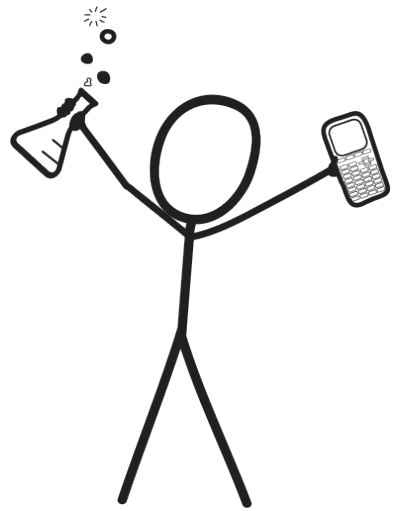
$\rightsquigarrow$ <u>New plan:</u> Get rid of the group, keep the graph.

Plan for this talk

- Some high-level intuition. ✓
- Elliptic curves & isogenies.
- Isogeny group actions.
- Signatures from isogenies.

Stand back!



We’re going to do math.

Elliptic curves

An elliptic curve over a field F of characteristic $\notin \{2, 3\}$ is* an equation of the form

$$E: \; y^2 = x^3 + ax + b$$

with $a, b \in F$ such that $4a^3 + 27b^2 \neq 0$.

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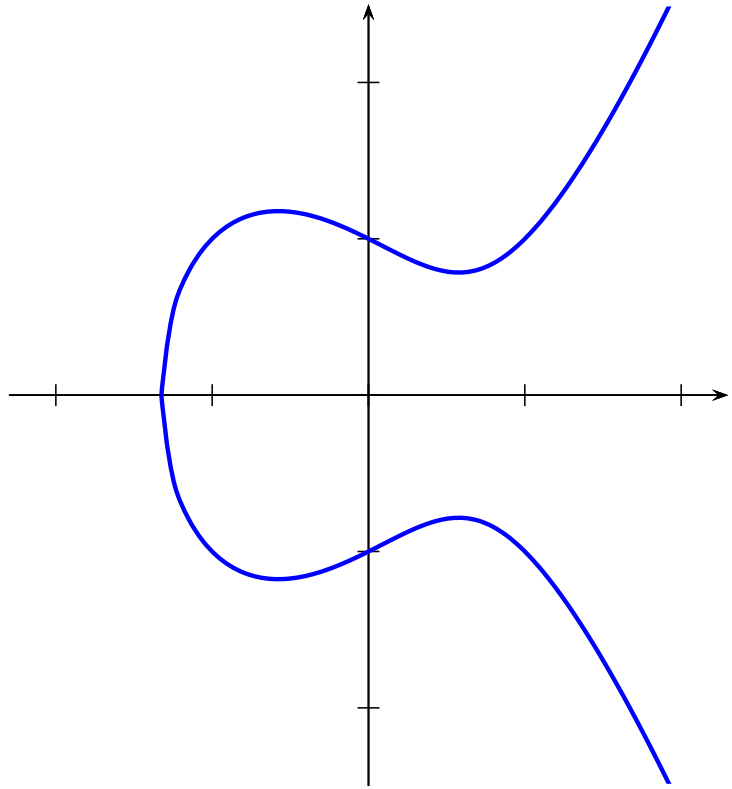
- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

$$\left(\lambda^2 - x_1 - x_2, \; \lambda(2x_1 + x_2 - \lambda^2) - y_1\right)$$

where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + a}{2y_1}$ otherwise.

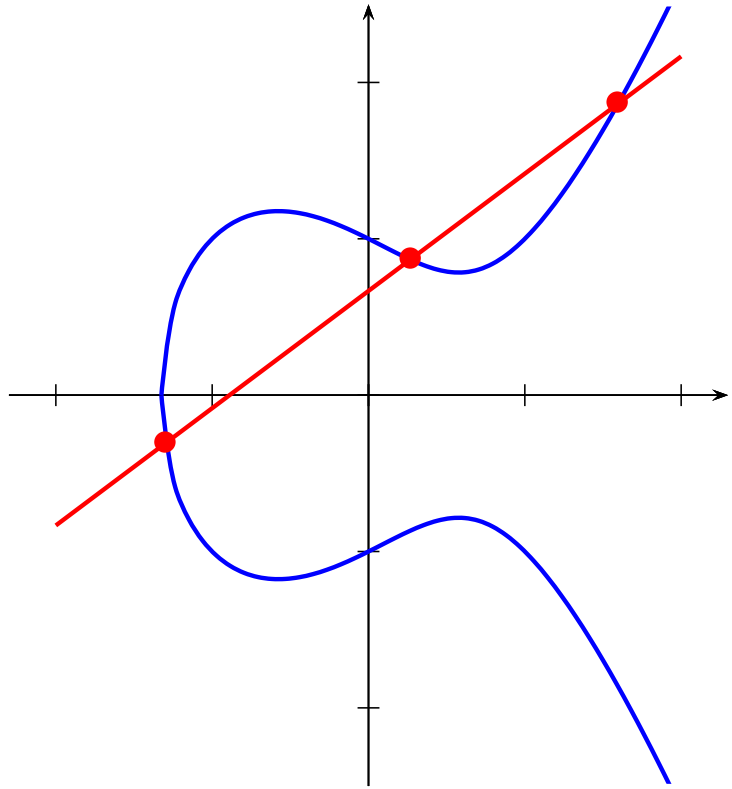
do **not** remember these formulas!

Elliptic curves (picture over $\mathbb{R}$)



The elliptic curve $y^2 = x^3 - x + 1$ over $\mathbb{R}$.

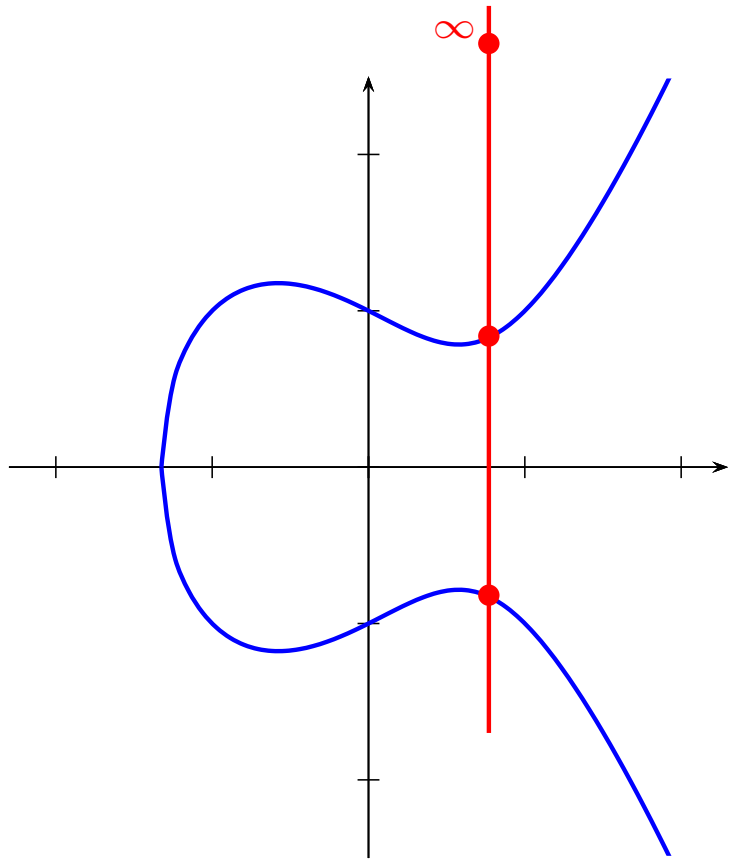
Elliptic curves (picture over $\mathbb{R}$)



Addition law:

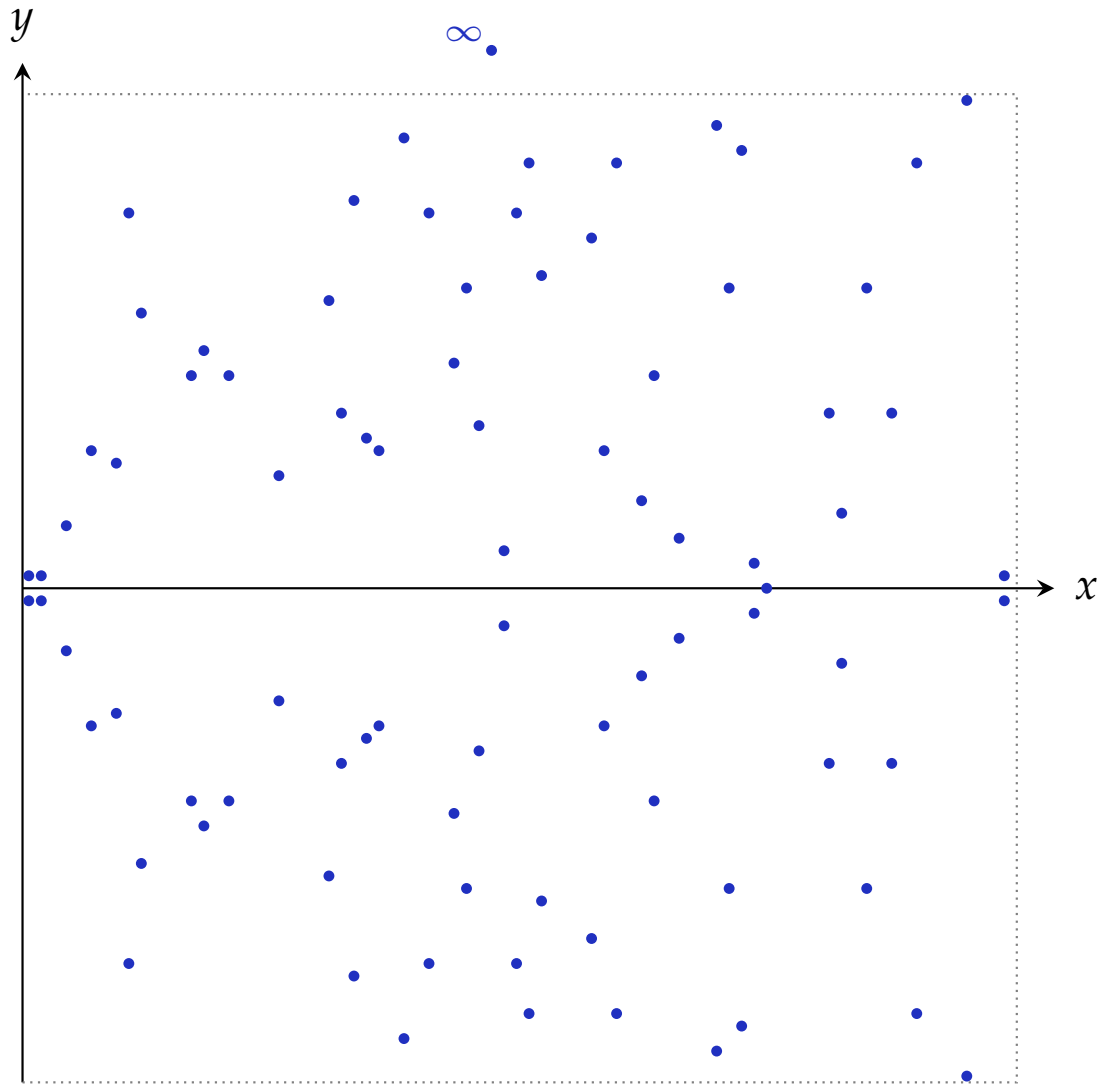
$$P + Q + R = \infty \iff \{P, Q, R\} \text{ on a straight line.}$$

Elliptic curves (picture over $\mathbb{R}$)



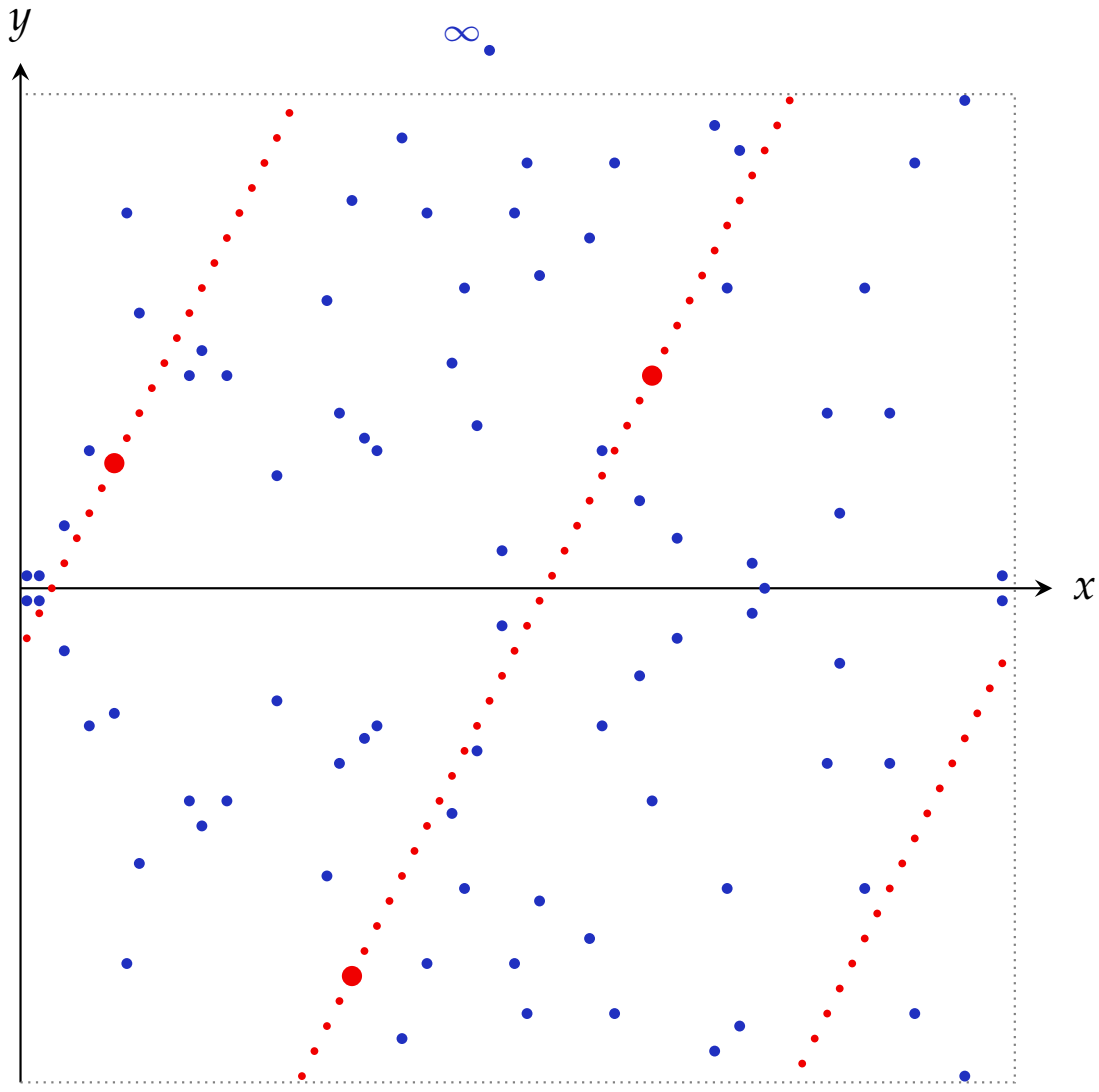
The *point at infinity* ∞ lies on every vertical line.

Elliptic curves (picture over $\mathbb{F}_p$)



The same curve $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

Elliptic curves (picture over $\mathbb{F}_p$)



The addition law of $y^2 = x^3 - x + 1$ over the finite field $\mathbb{F}_{79}$.

ECDH (not post-quantum)

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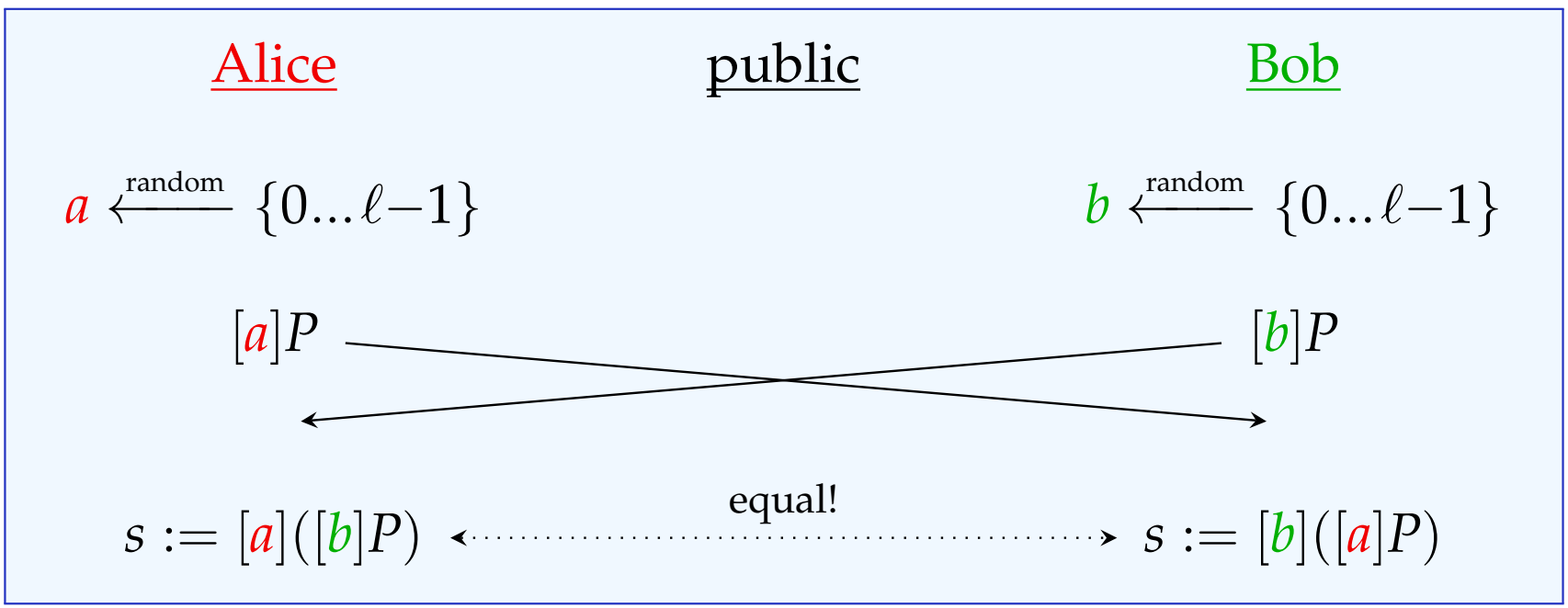

Isogenies

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...are just fancily-named

nice maps

between elliptic curves.

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A rational function is $f(x,y)/g(x,y)$ where f, g are polynomials.

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The kernel of an isogeny $\varphi\colon E \to E'$ is $\{P \in E \,:\, \varphi(P) = \infty\}$.

The degree of a separable* isogeny is the size of its kernel.

(This matches the degree of φ in x when written in lowest terms.)

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Example #1: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$ defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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Example #2: For each $m \neq 0$, the multiplication-by-m map

$$[m] : E \to E$$

is a degree-m^2 isogeny. Notation: $E[m] := \ker[m]$.

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Important fact: An isogeny φ is $\mathbb{F}_q$-rational iff $\pi \circ \varphi = \varphi \circ \pi$.

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Tate's theorem:
$E, E'/\mathbb{F}_q$ are isogenous over $\mathbb{F}_q$ if and only if $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$.

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$\Longrightarrow$ Bottom line: Being isogenous is an equivalence relation.
Over finite fields, we can easily test it.

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$\rightsquigarrow$ Decompose large-degree isogenies into prime steps.
That is, walk in an isogeny graph.

Computing isogenies: Vélu's formulas (1971)

Let G be a finite subgroup of an elliptic curve E. Then

$$P \mapsto \left(x(P) + \sum_{Q \in G \setminus \{\infty\}} \big(x(P+Q) - x(Q)\big), \right.$$
$$\left. y(P) + \sum_{Q \in G \setminus \{\infty\}} \big(y(P+Q) - y(Q)\big) \right)$$

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This leads to formulas for

- computing the defining equation of E/G;
- evaluating the isogeny $E \to E/G$ at a point.

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defines an isogeny of elliptic curves with kernel G.

This leads to formulas for

- computing the defining equation of E/G;
- evaluating the isogeny $E \to E/G$ at a point.

Complexity: $\Theta(\#G) \rightsquigarrow$ only suitable for small degrees.

The $\sqrt{\text{élu}}$ algorithm reduces the cost to $\widetilde{\mathcal{O}}(\sqrt{\#G})$.

Predictable groups

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- In that case, $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$ or $E(\mathbb{F}_p) \cong \mathbb{Z}/\frac{p+1)}{2} \times \mathbb{Z}/2$, and $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}/(p+1) \times \mathbb{Z}/(p+1)$.

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$\rightsquigarrow$ Easy method to control the group structure by choosing p!

$\rightsquigarrow$ Cryptography works well using supersingular curves.

(All curves are supersingular for the rest of the lecture.)

“Computing an isogeny”

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Keep in mind: Constructing isogenies $E \to \_$ is (usually) easy, constructing an isogeny $E \to E'$ given (E, E') is (usually) hard.

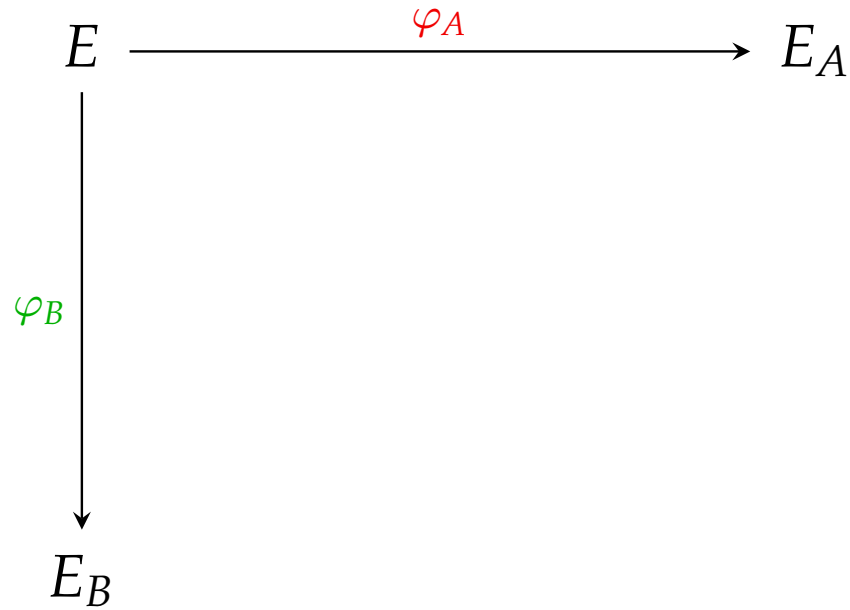
Plan for this talk

- Some high-level intuition. ✓
- Elliptic curves & isogenies. ✓
- Isogeny group actions.
- Signatures from isogenies.

Isogeny-based key exchange: High-level view

E

Isogeny-based key exchange: High-level view



- Alice & Bob pick secret $\varphi_A : E \to E_A$ and $\varphi_B : E \to E_B$.
  (These isogenies correspond to walking on the isogeny graph.)

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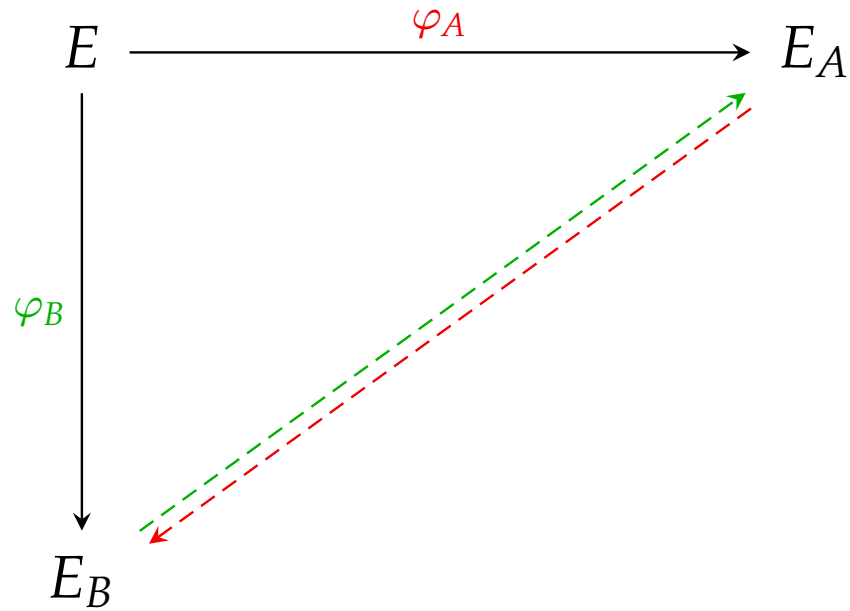

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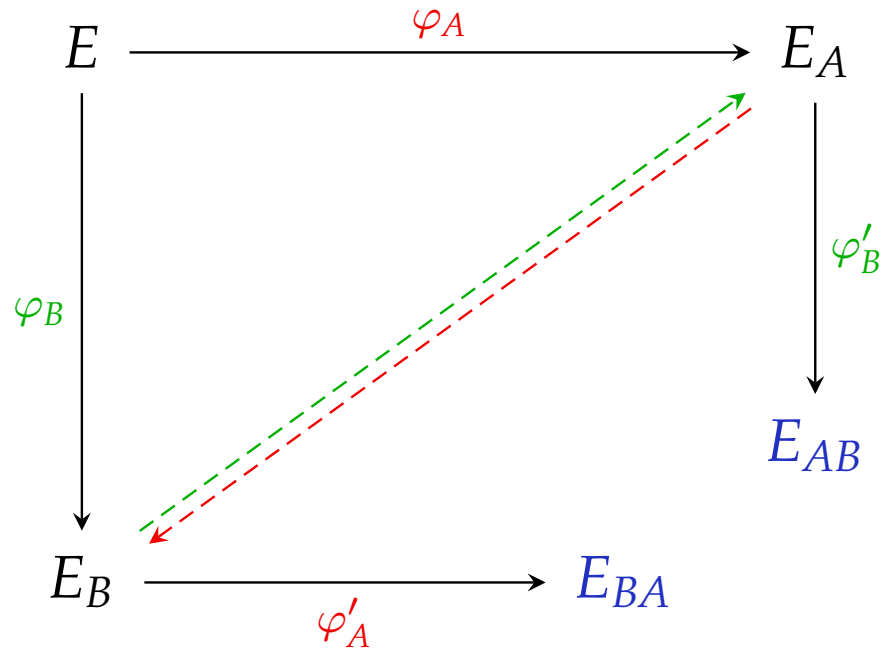

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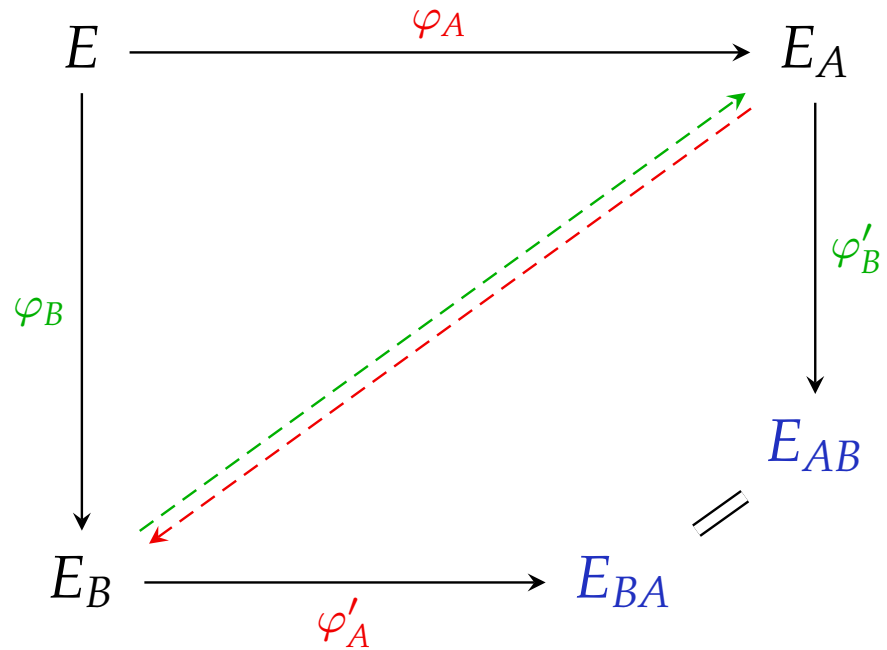

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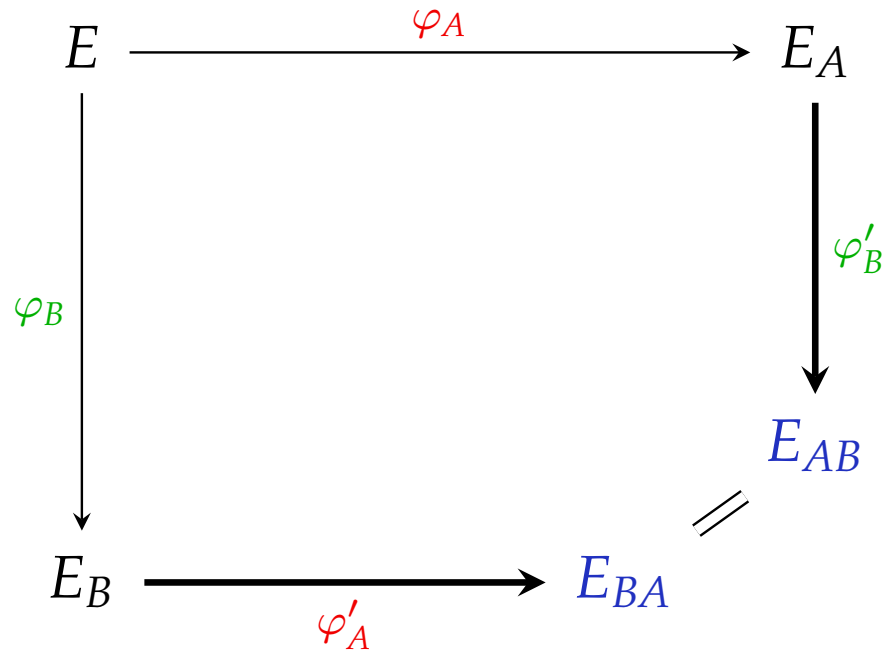
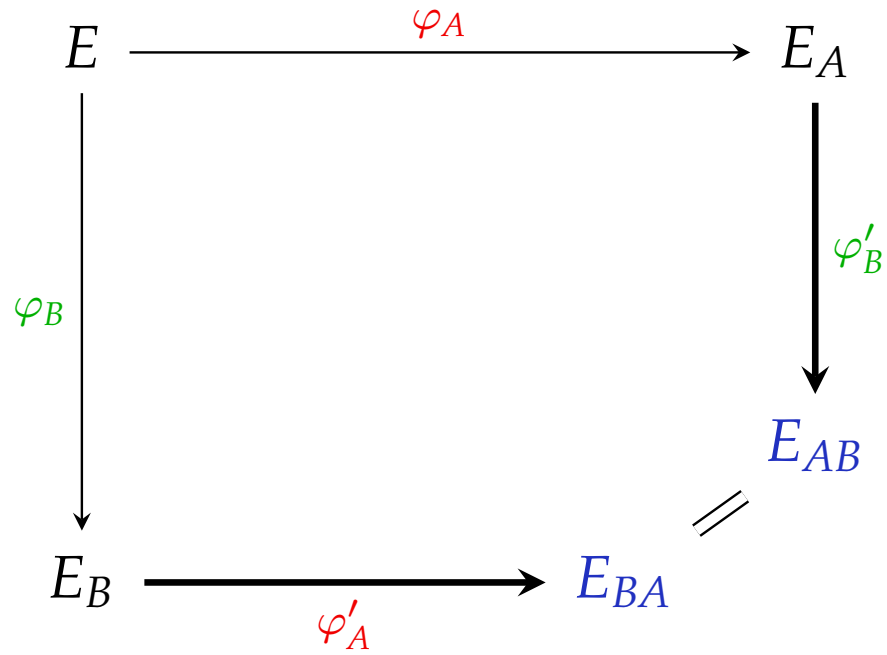
- Alice and Bob transmit the end curves E_A and E_B.
- Alice somehow finds a "parallel" $\varphi_{A'} : E_B \to E_{BA}$, and Bob somehow finds $\varphi_{B'} : E_A \to E_{AB}$, such that $E_{AB} \cong E_{BA}$.

How to find “parallel” isogenies?



How to find "parallel" isogenies?



CSIDH's solution:

Use special isogenies φ_A which can be transported to the curve E_B totally independently of the secret isogeny φ_B.

(Similarly with reversed roles, of course.)


CSIDH ['si:ˌsaɪd]
[Castryck–Lange–Martindale–Panny–Renes 2018]

“Special” isogenies

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We consider prime ℓ and refer to φ_ℓ as a "special" isogeny.

Cycles from “special” isogenies

What happens when we iterate such a “special” isogeny?

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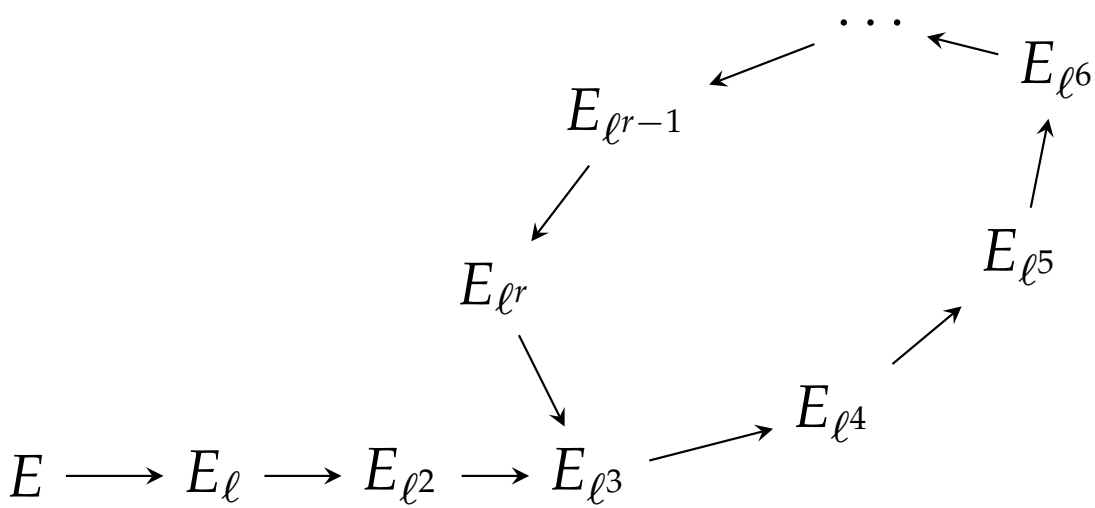

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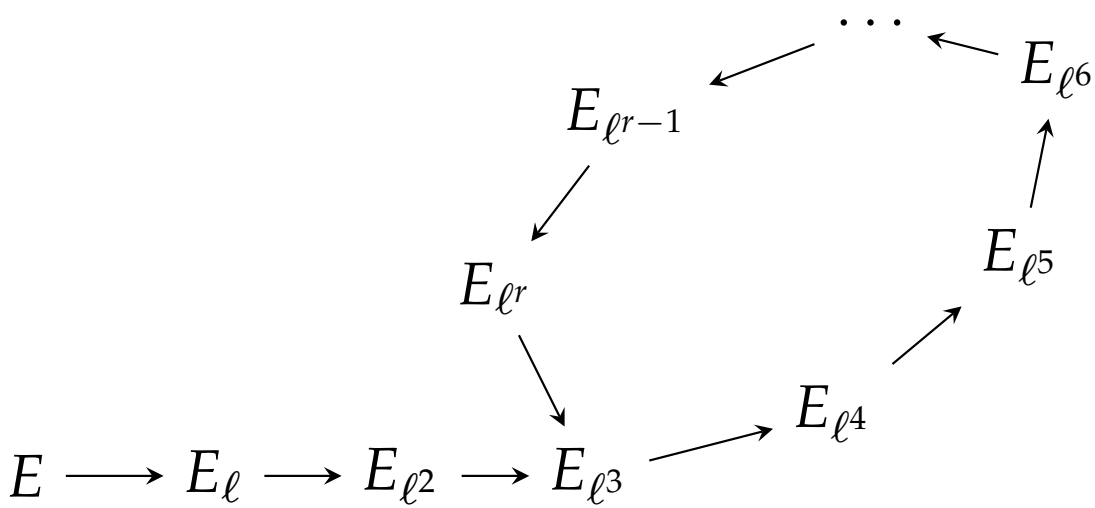

- Fact: Each curve has only one other rational ℓ-isogeny.
  It is defined by the kernel $\{P \in E(\mathbb{F}_{p^2}) : [\ell_i]P = \infty \wedge \pi(P) = -P\}$.

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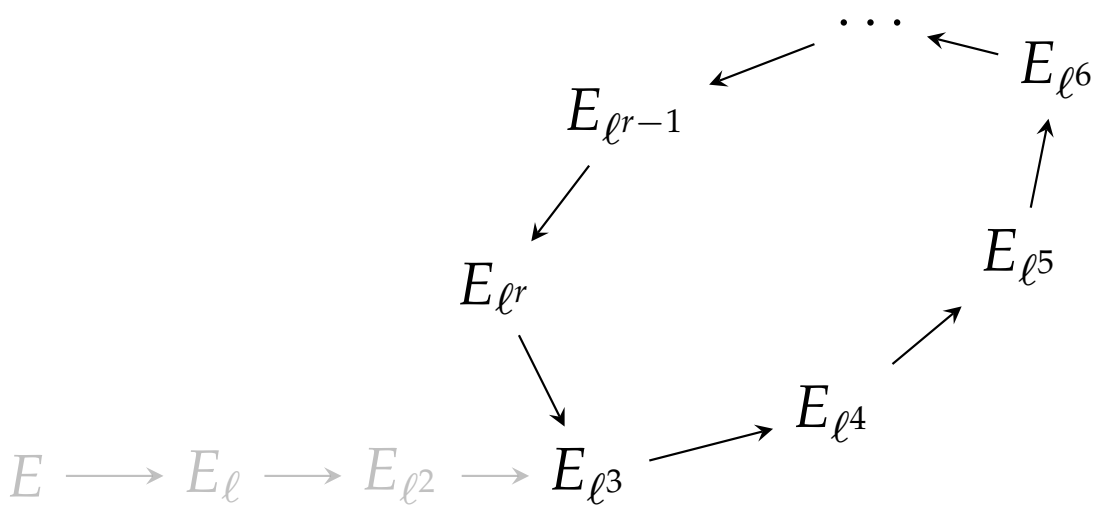

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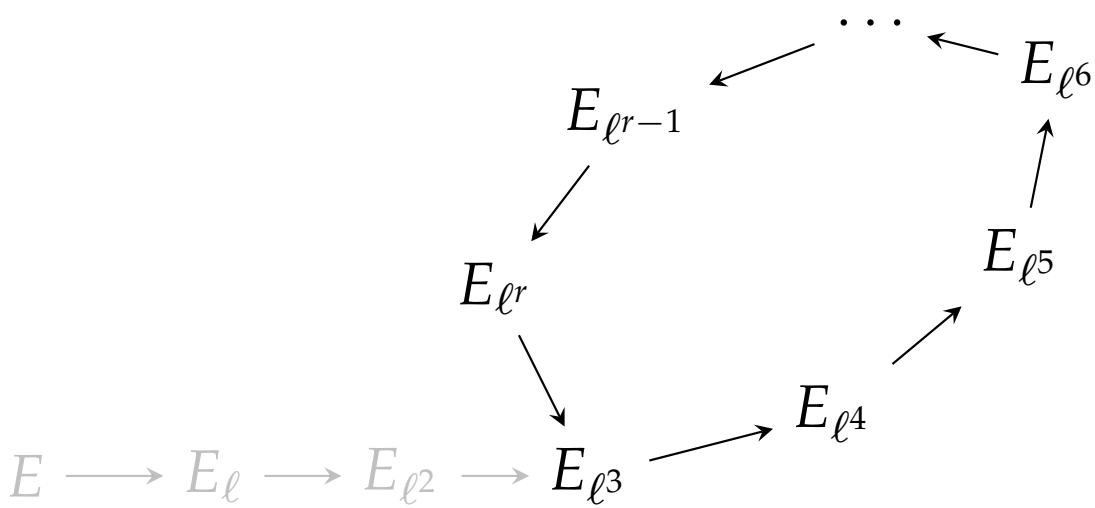

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$\Longrightarrow$ The "special" isogenies φ_ℓ form isogeny cycles!

Compatible cycles from "special" isogenies

What happens when we compose those "special" isogenies?

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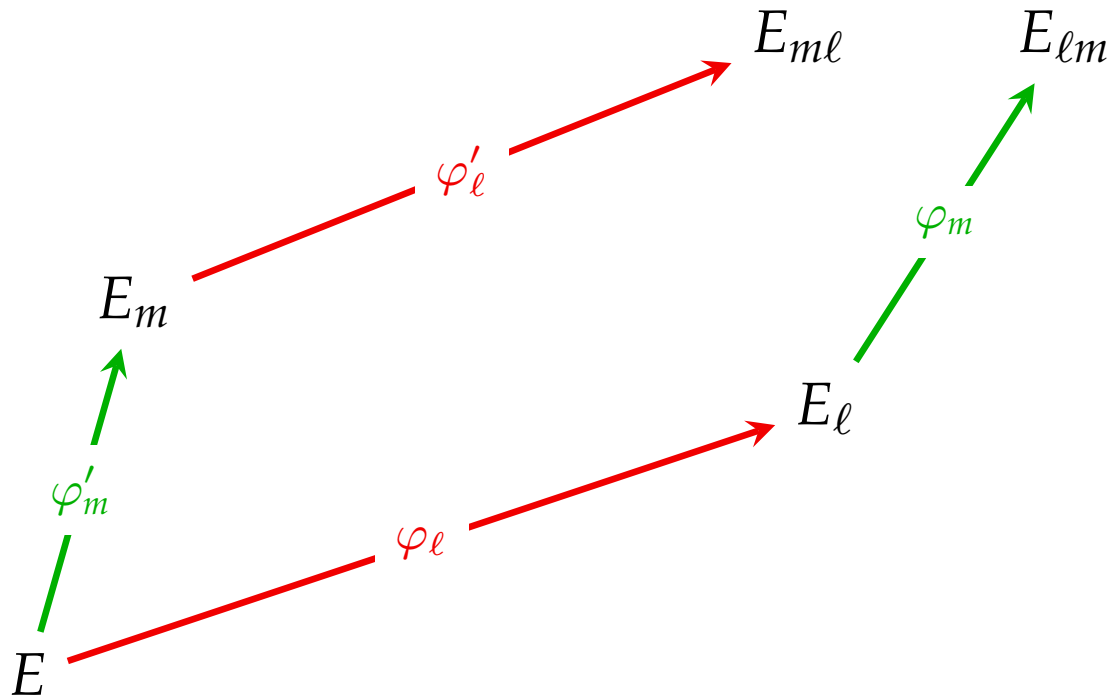

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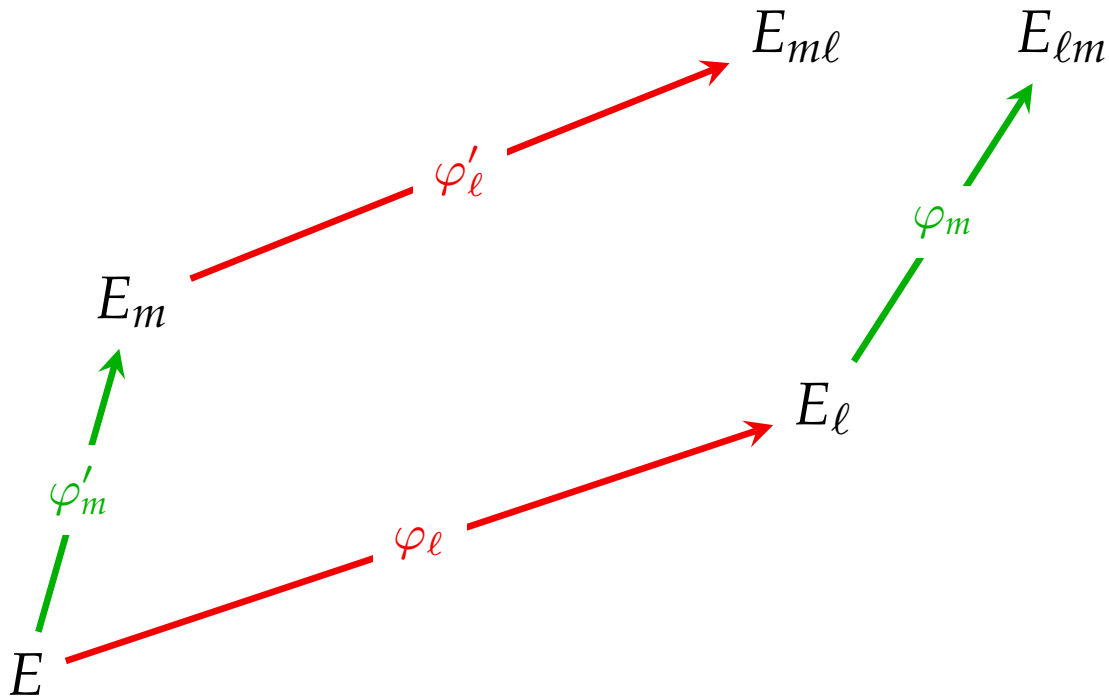

- Fact: $\ker(\varphi'_\ell \circ \varphi'_m) = \ker(\varphi_m \circ \varphi_\ell) = \langle \ker \varphi_\ell, \ker \varphi'_m \rangle$.

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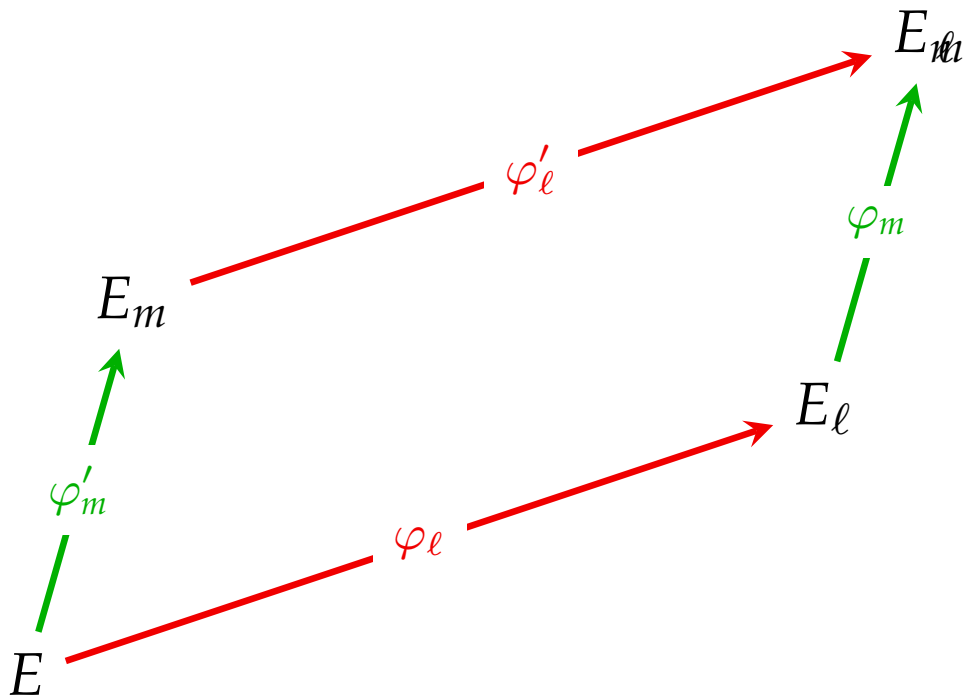

- Fact: $\ker(\varphi'_\ell \circ \varphi'_m) = \ker(\varphi_m \circ \varphi_\ell) = \langle \ker \varphi_\ell, \ker \varphi'_m \rangle$.

!! The order cannot matter $\Longrightarrow$ cycles must be compatible.

CSIDH in one slide

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- Choose some small odd primes $\ell_1, \ldots, \ell_n$.
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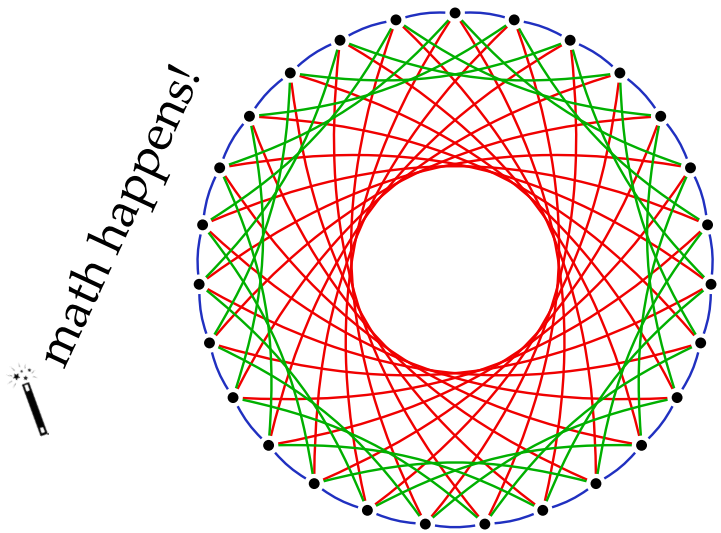

$p = 419$
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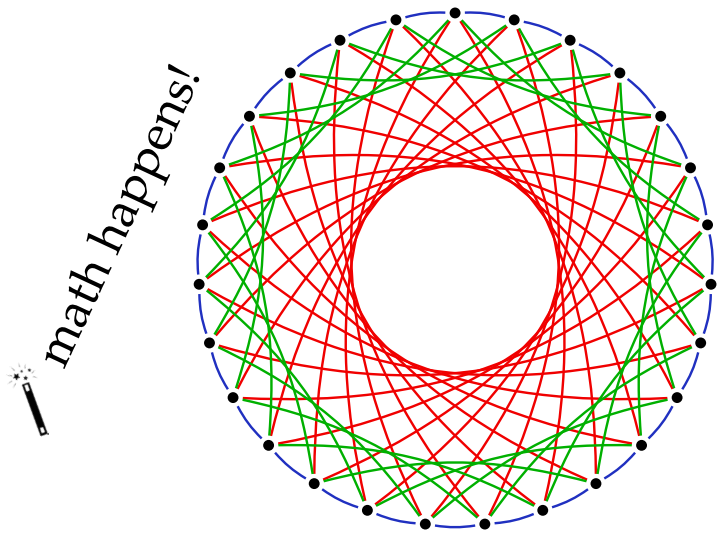

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- Walking "left" and "right" on any ℓ_i-subgraph is efficient.

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- Recall: "Left" and "right" steps correspond to isogenies with special subgroups of E as kernels.

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(Finding a point of order ℓ_i: Pick $x \in \mathbb{F}_p$ random. Find $y \in \mathbb{F}_{p^2}$ such that $P = (x, y) \in E$. Compute $Q = [\frac{p+1}{\ell_i}]P$. Hope that $Q \neq \infty$, else retry.)

Walking in the CSIDH graph (in SageMath)

```
sage: E = EllipticCurve(GF(419^2), [1,0])
sage: E
Elliptic Curve defined by y^2 = x^3 + x
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sage: P
(218 : 403 : 1)
sage: P.order().factor()
2 * 3 * 7
sage: EE = E.isogeny_codomain(2*3*P)  # "left" 7-step
sage: EE
Elliptic Curve defined by y^2 = x^3 + 285*x + 87
        over Finite Field in z2 of size 419^2
```
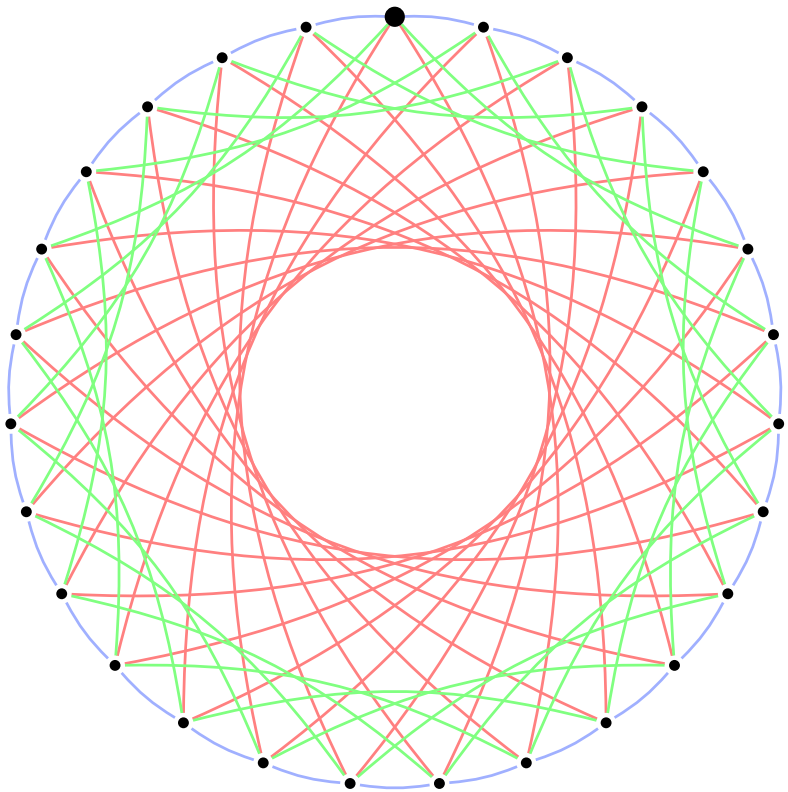

CSIDH key exchange

Alice

$[+, +, -, -]$



Bob

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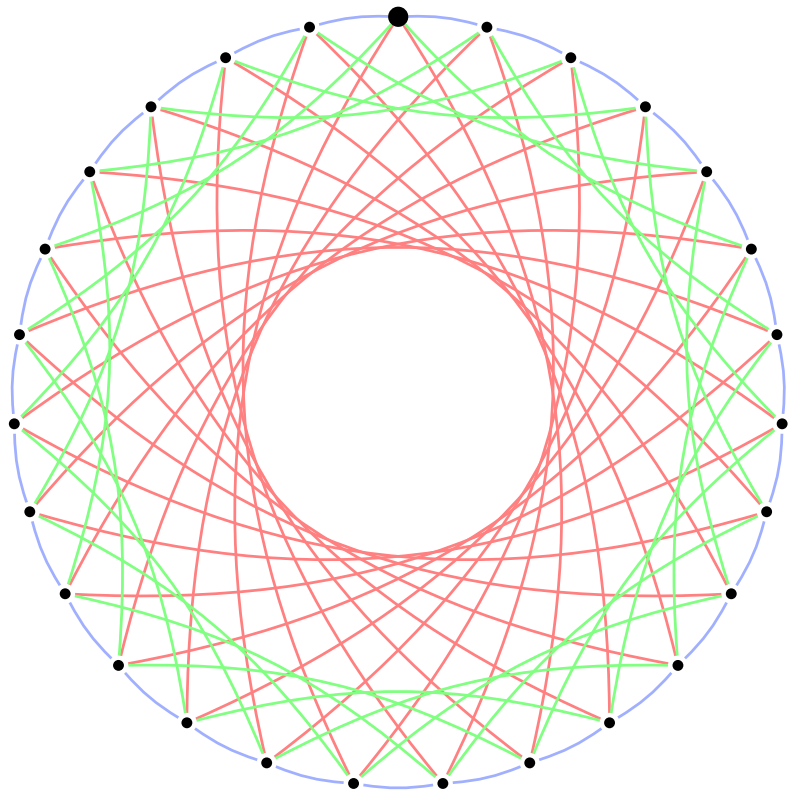

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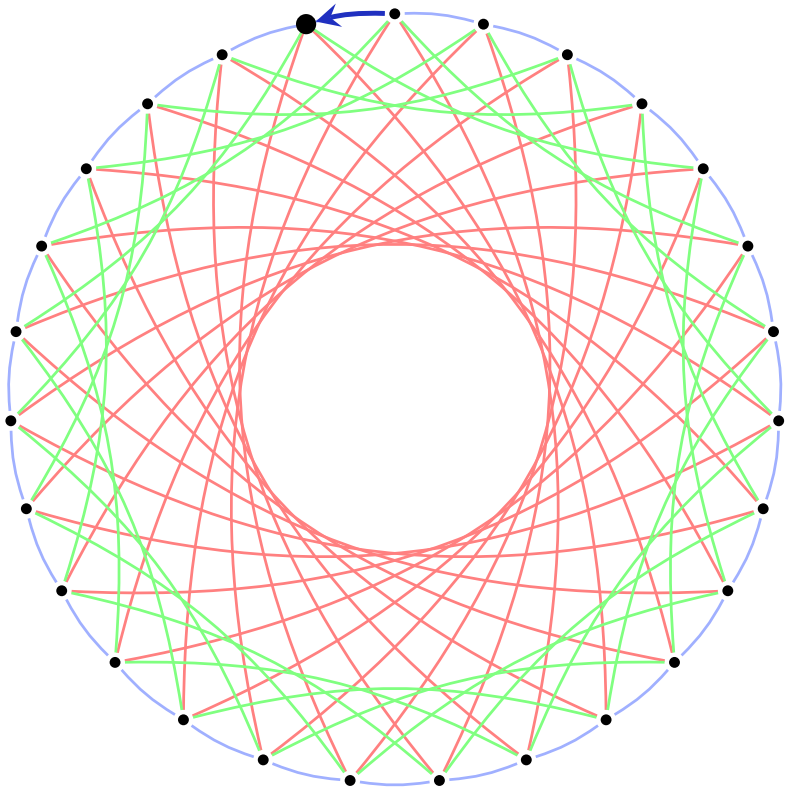

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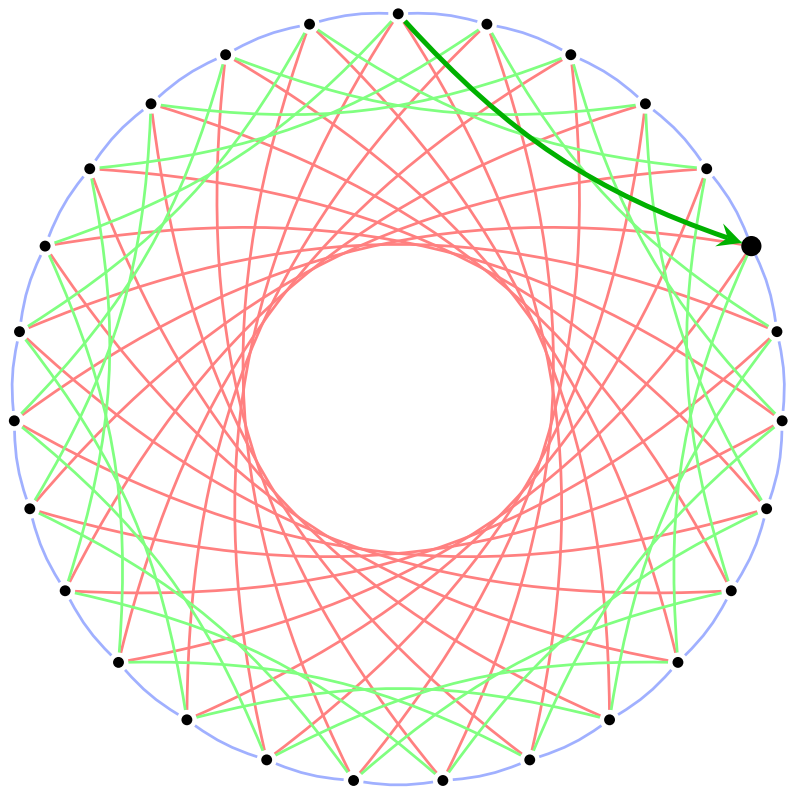

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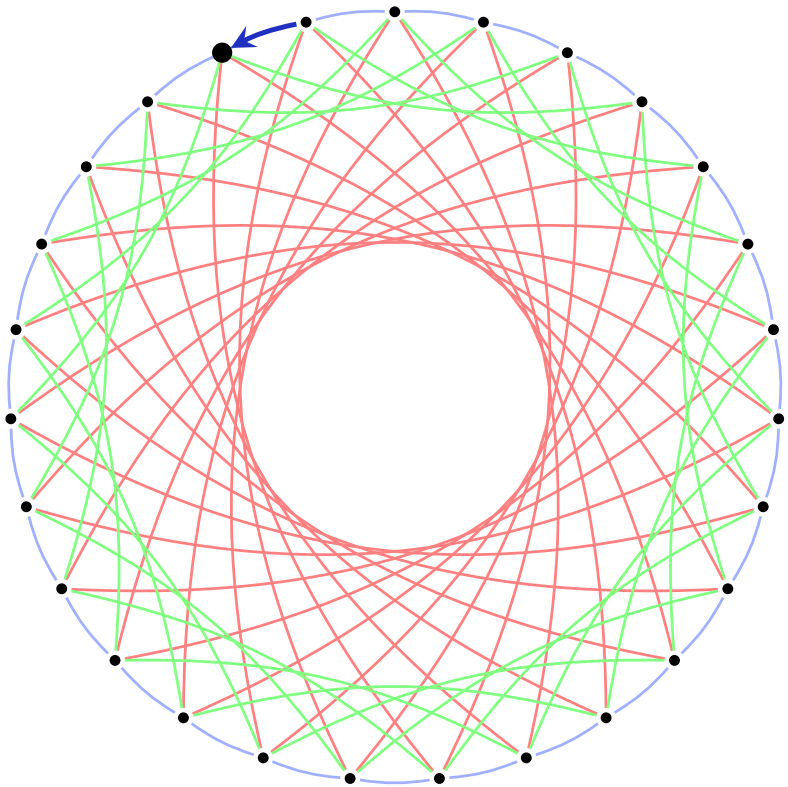

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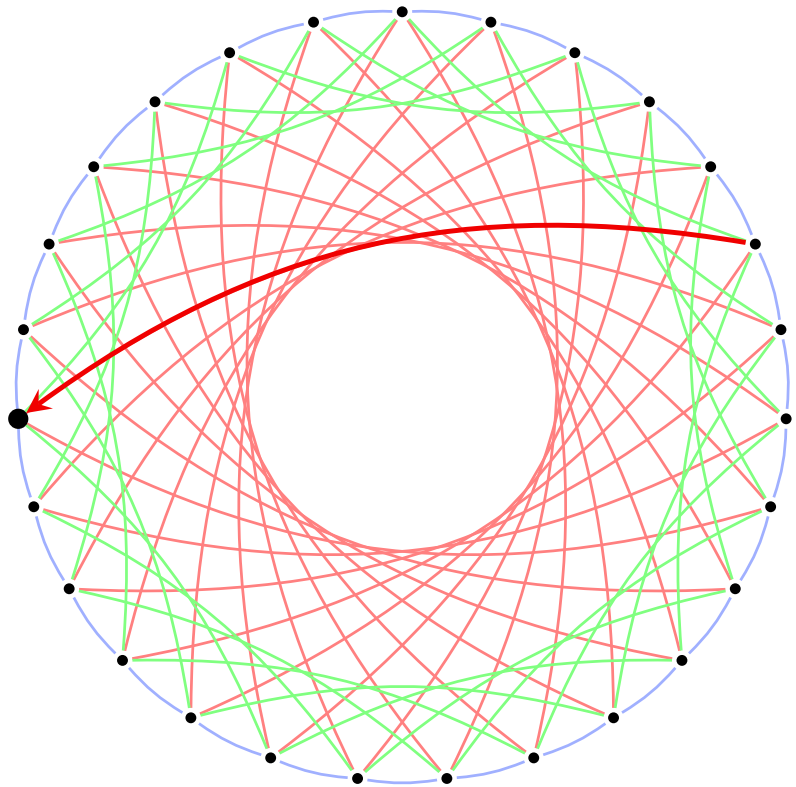

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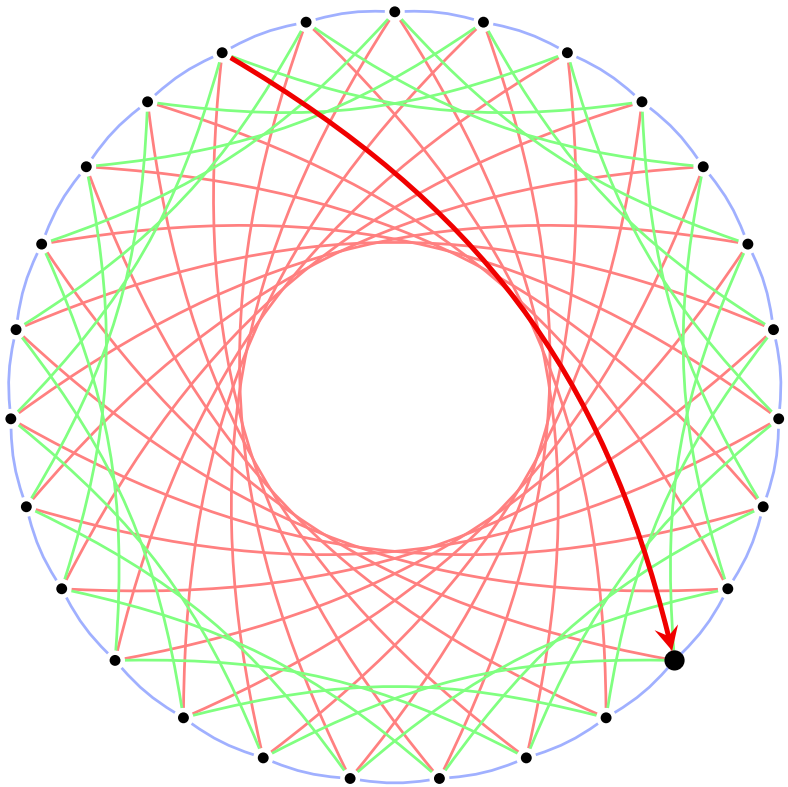

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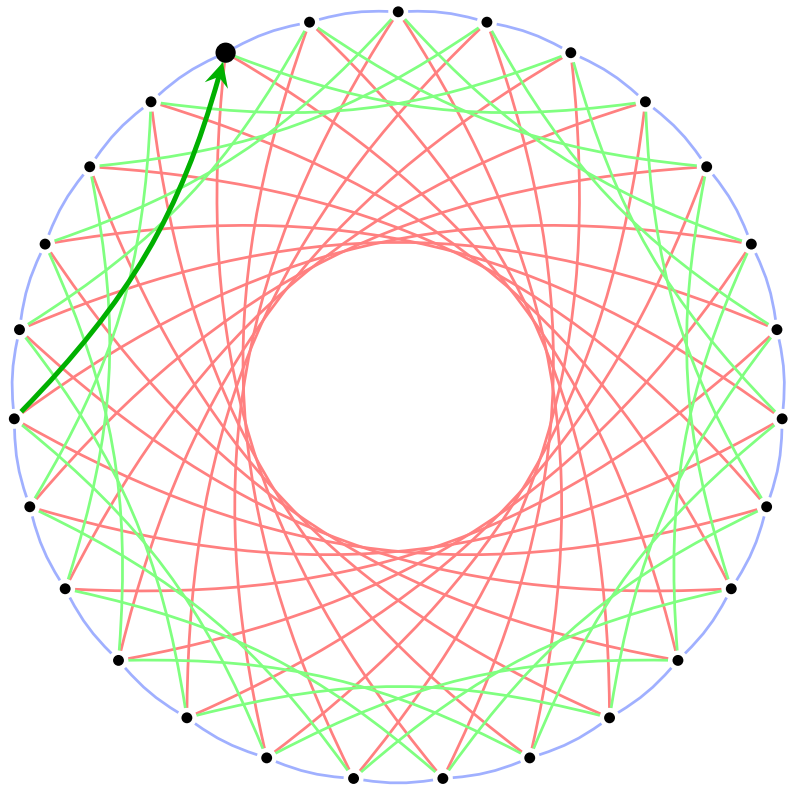

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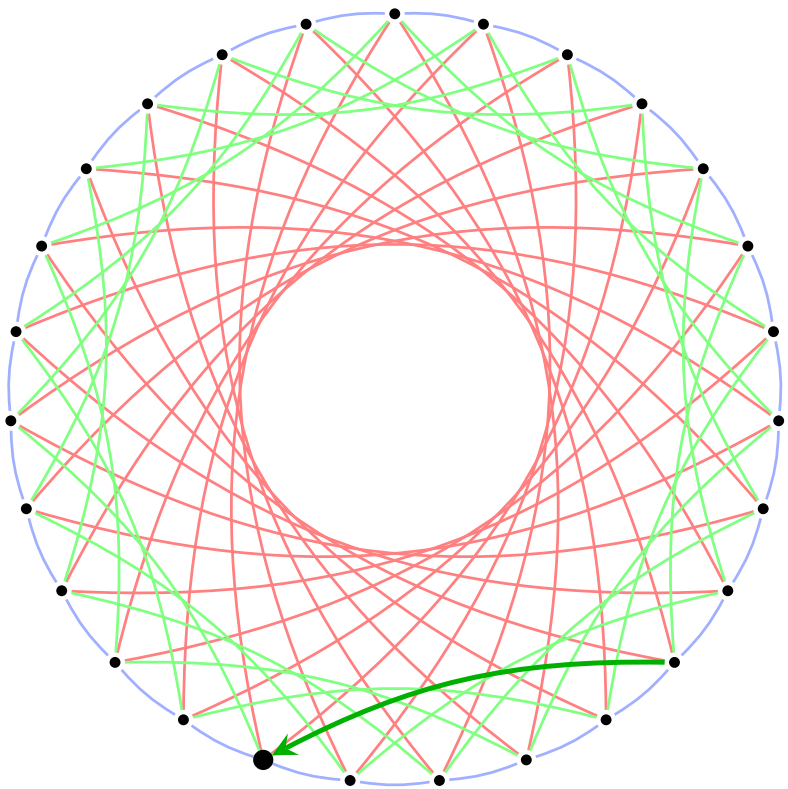

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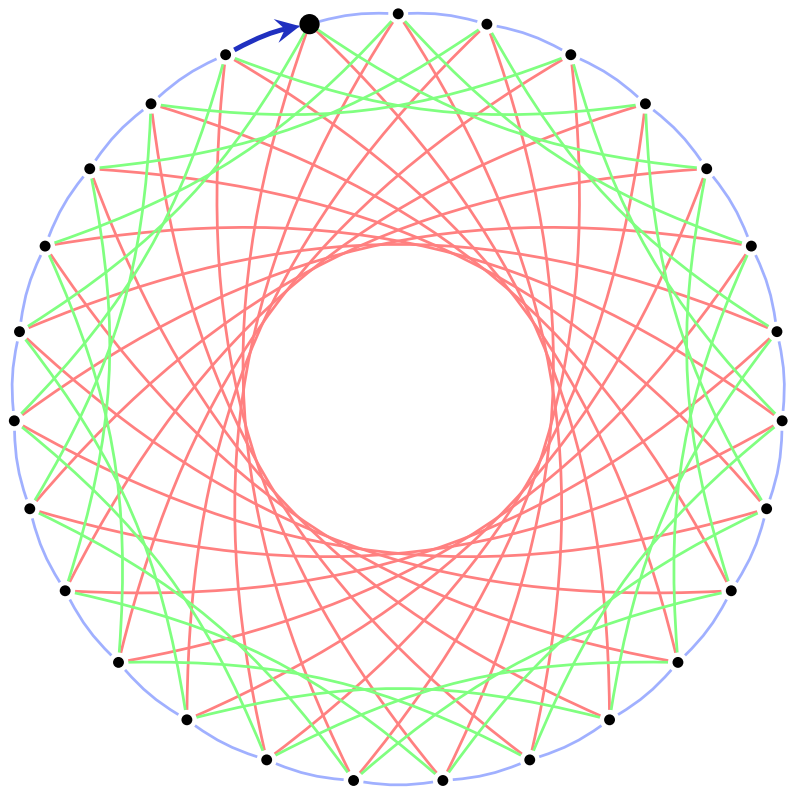

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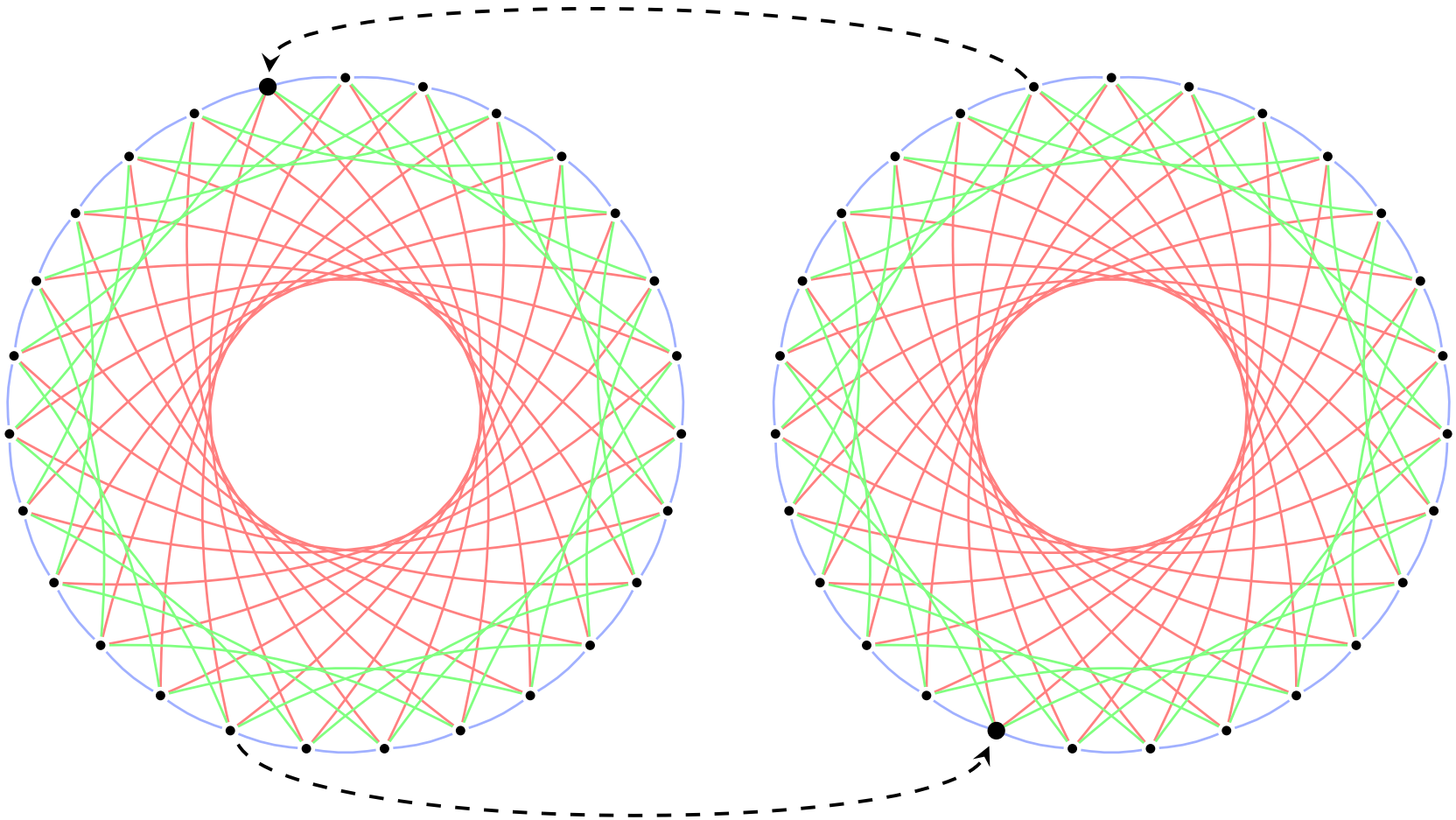

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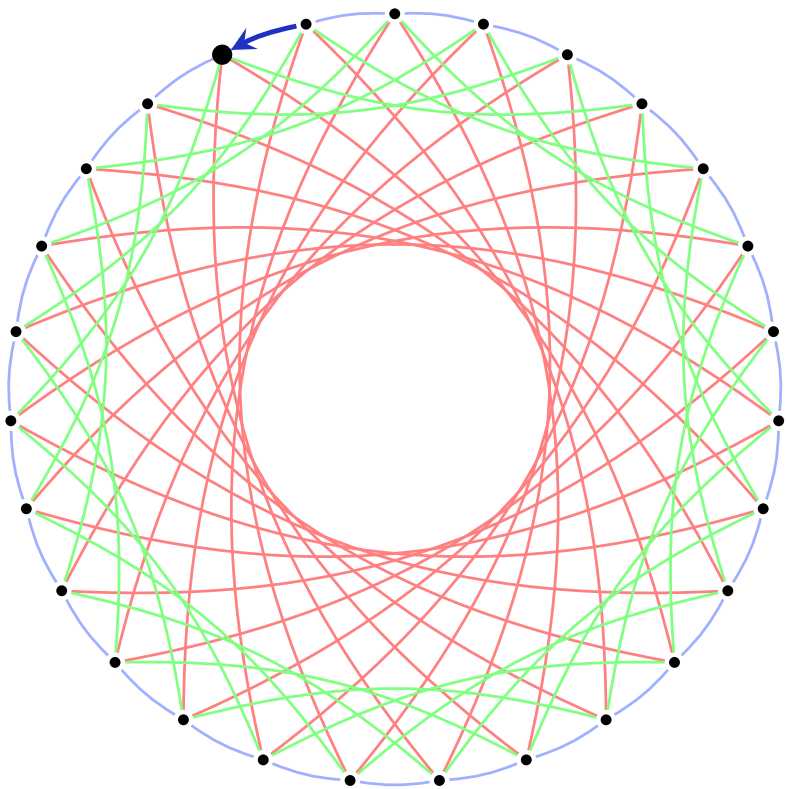

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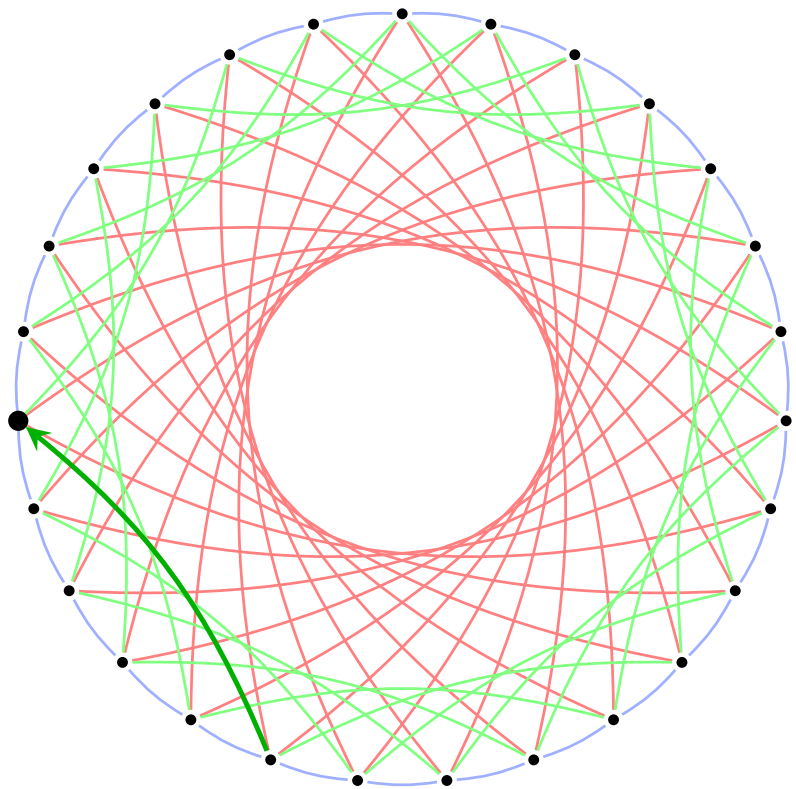

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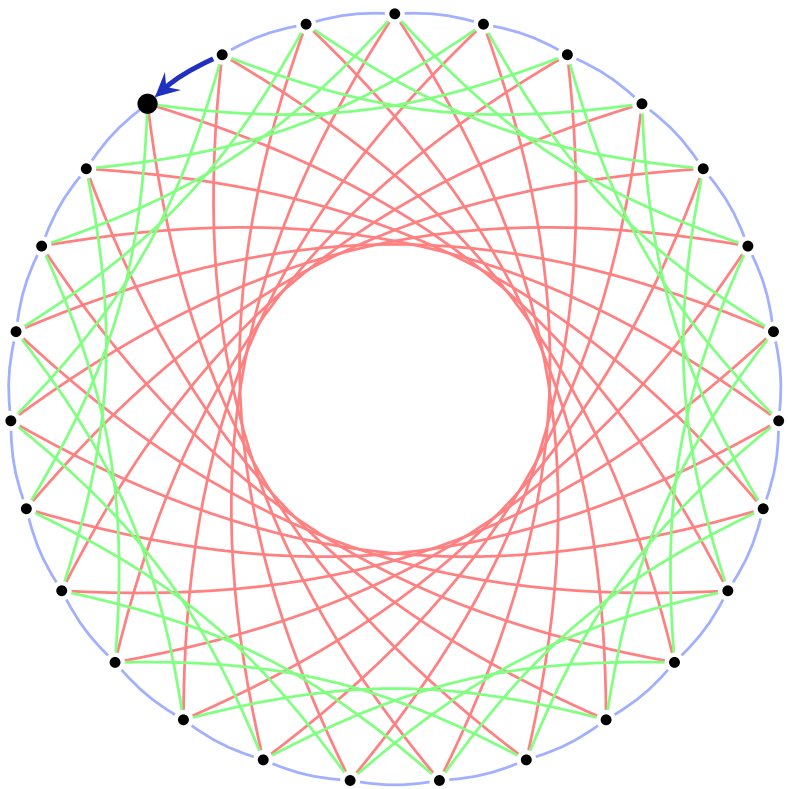

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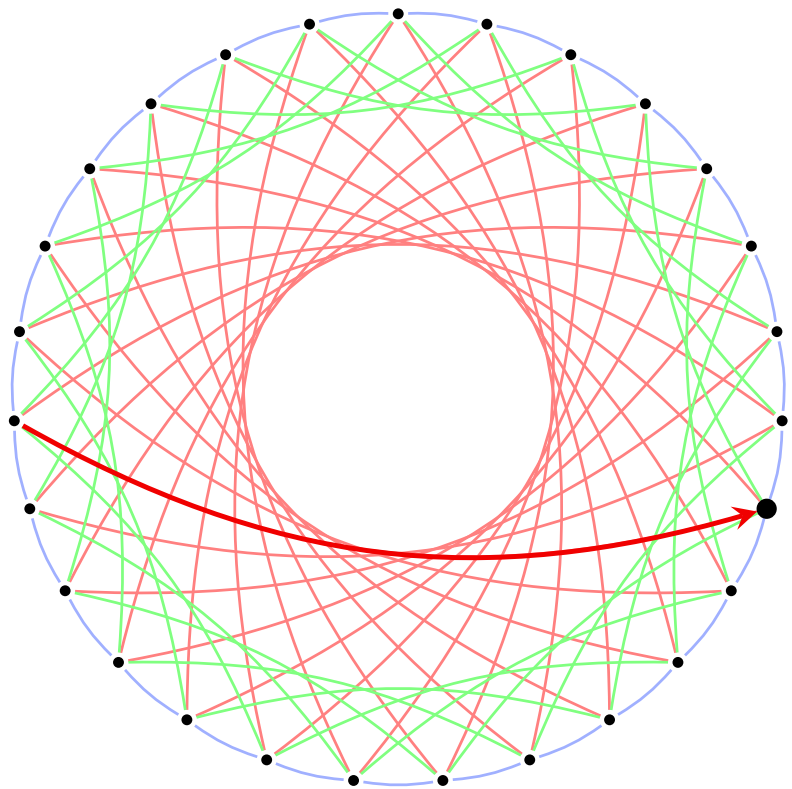

CSIDH key exchange

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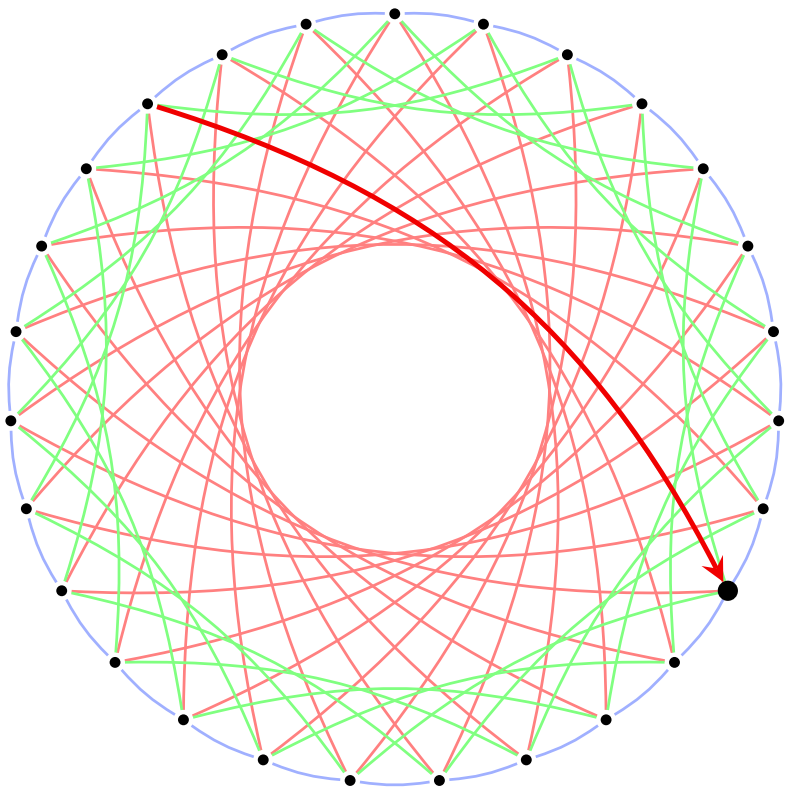

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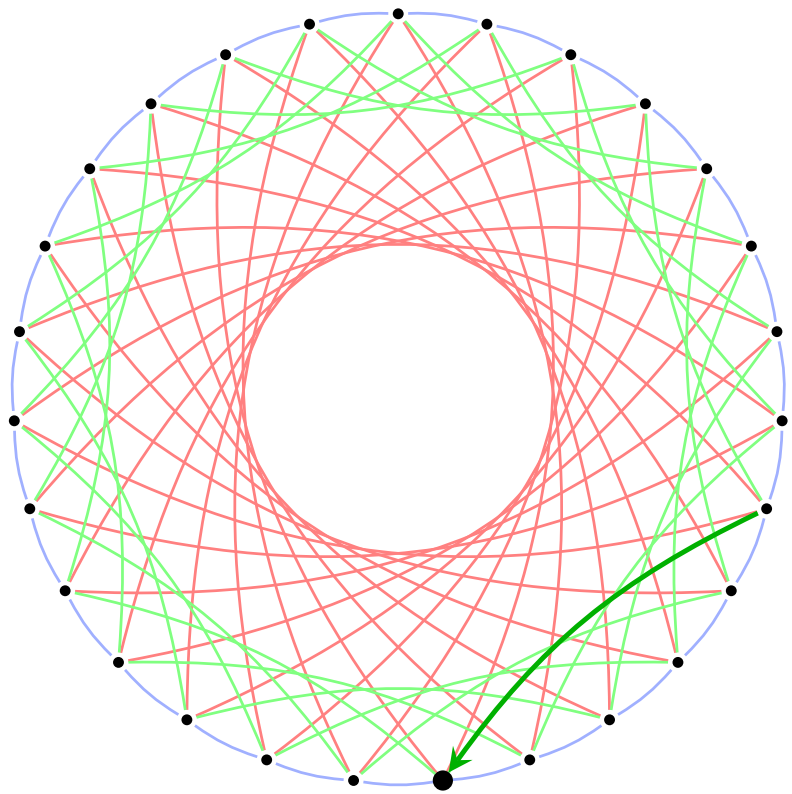

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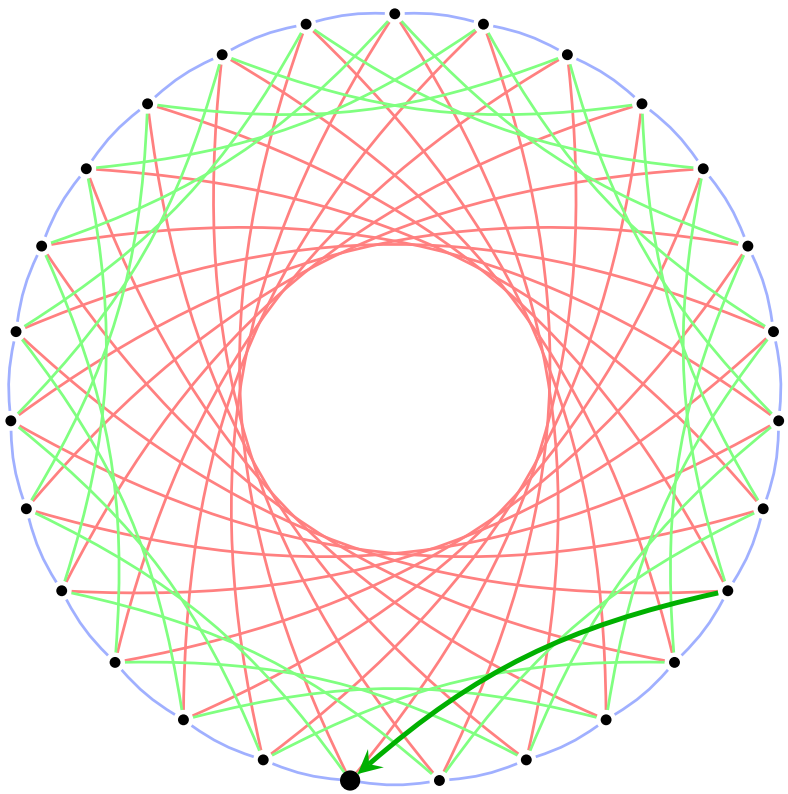

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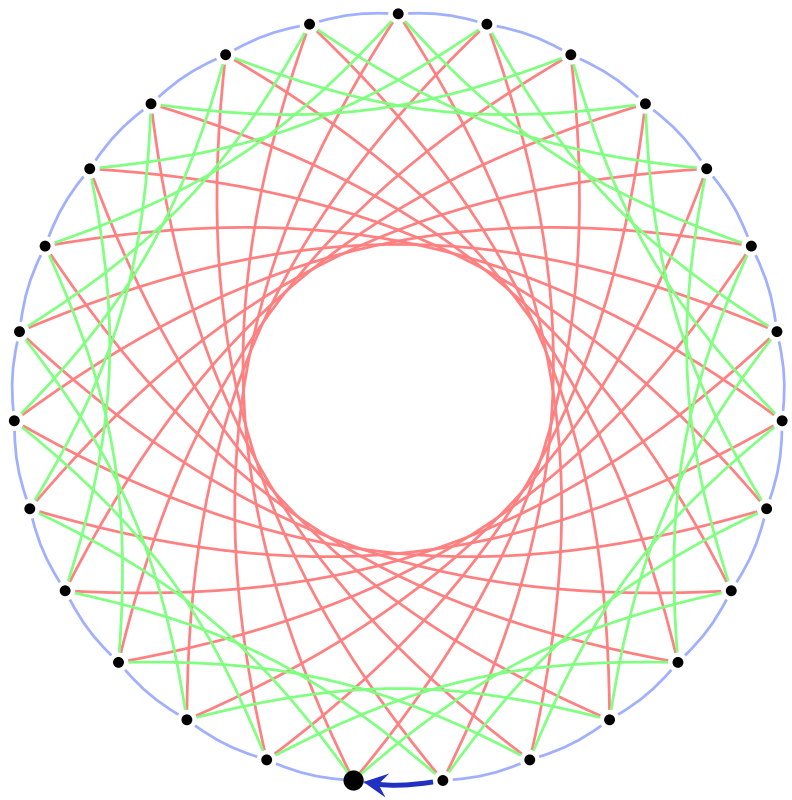

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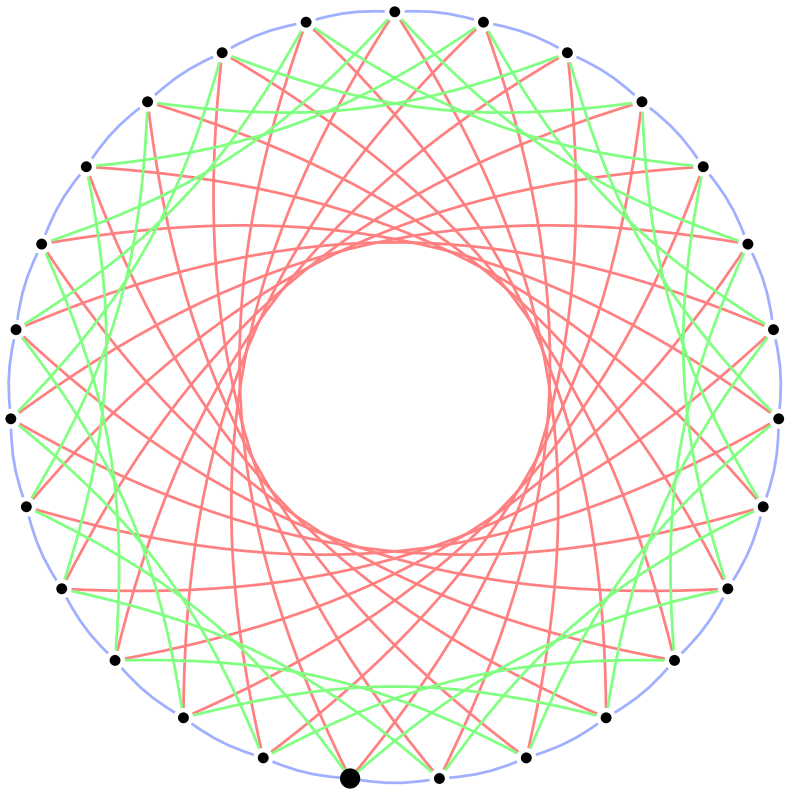

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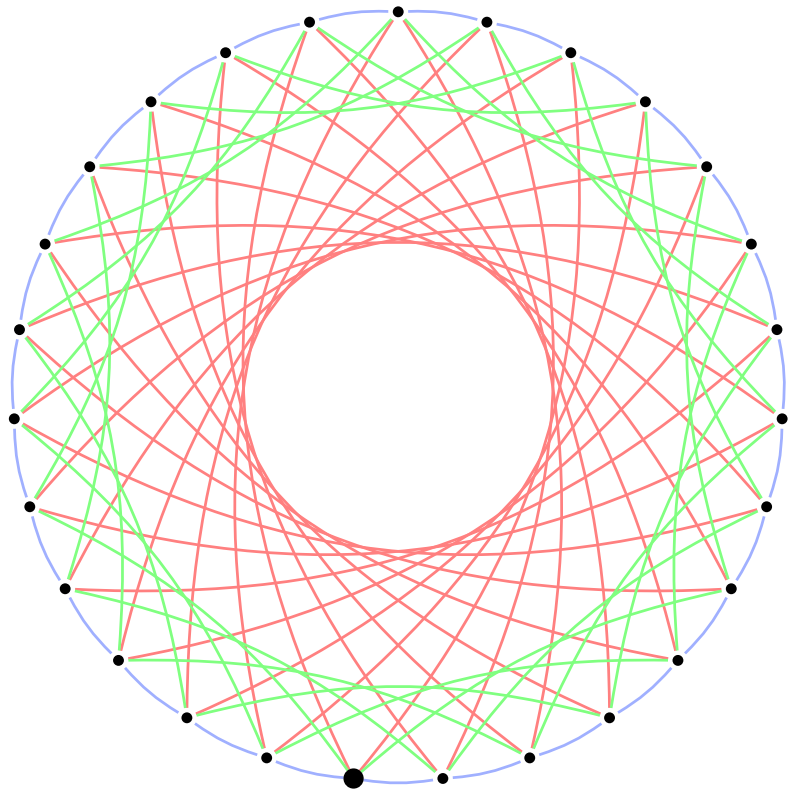

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(An action of a group $(G, \cdot)$ on a set X is a map $*\colon G \times X \to X$
such that $id * x = x$ and $g * (h * x) = (g \cdot h) * x$ for all $g, h \in G$ and $x \in X$.)

The class group

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The lattice Λ is computable in subexponential time classically, and in polynomial time using a quantum computer.
It is used to construct more advanced schemes (*"CSI-FiSh"*).

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For group actions, we simply cannot compose $a * s$ and $b * s$!

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Core problem:
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Solving abelian hidden shift breaks CSIDH.

$\rightsquigarrow$ non-devastating quantum attack (Kuperberg's algorithm).

Subexponential: Complexity $\exp\big((\log p)^{1/2+o(1)}\big)$.

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Kuperberg's algorithm consists of two components:

1. Evaluate the group action many times. ("oracle calls")
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$\Longrightarrow$ Security estimates for CSIDH & friends vary wildly. 

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- 2023: "Clapoti" — a polynomial-time algorithm for arbitrary combinations of operations in the group and evaluations of the action. $\rightsquigarrow$ *"KLaPoTi"*, *"PEGASIS"*.
(Previously, only restricted sequences of operations were efficient.)

$\rightsquigarrow$ *Friday.* ☺

Other isogeny group actions

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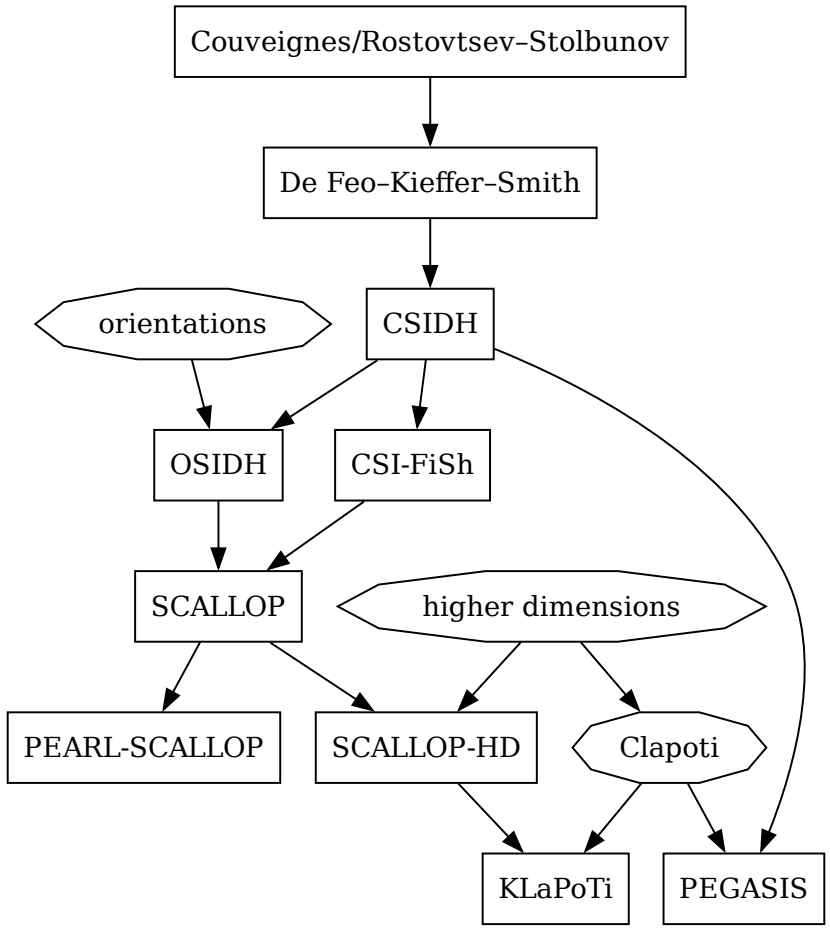

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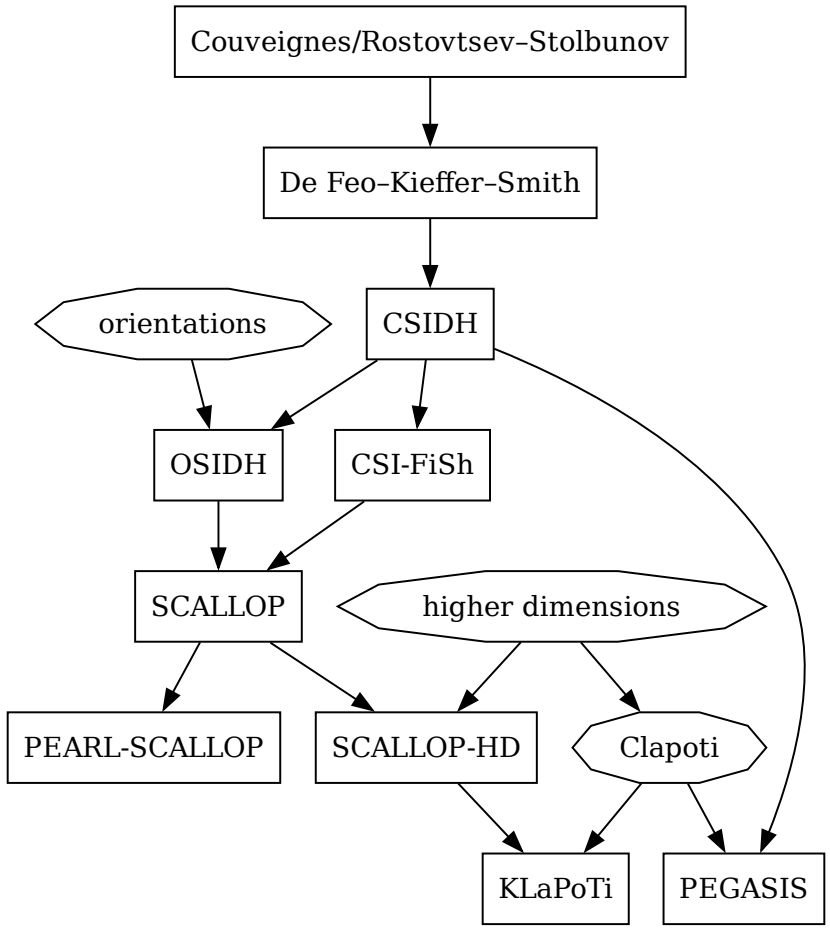

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Plan for this talk

- Some high-level intuition. ✓
- Elliptic curves & isogenies. ✓
- Isogeny group actions. ✓
- Signatures from isogenies.

SQIsign: What?

https://sqisign.org

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- A new-ish and very hot post-quantum signature scheme.
- Based on super cool mathematics. ◡̈

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☺ We understand how I_φ, I_ψ relate for isogenies $\varphi, \psi\colon E \to E'$.

$\rightsquigarrow$ one-sided ideal class *set* of $\mathrm{End}(E)$, etc.

The Deuring correspondence

...is the formal version of what I just said.

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Theorem. Fix E_0 supersingular. The (contravariant) functor

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☺ One direction is easy, the other seems hard! $\rightsquigarrow$ *Cryptography!*

The Deuring correspondence (examples)

Let $p = 7799999$ and let $\mathbf{i}, \mathbf{j}$ satisfy $\mathbf{i}^2 = -1$, $\mathbf{j}^2 = -p$, $\mathbf{ji} = -\mathbf{ij}$.

The ring $\mathcal{O}_0 = \mathbb{Z} \oplus \mathbb{Z}\,\mathbf{i} \oplus \mathbb{Z}\,\frac{\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{1+\mathbf{ij}}{2}$
corresponds to the curve $E_0 : \; y^2 = x^3 + x$.

The ring $\mathcal{O}_1 = \mathbb{Z} \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+32631010\mathbf{i}+\mathbf{ij}}{9894}$
corresponds to the curve $E_1 : \; y^2 = x^3 + 1$.

The ideal $I = \mathbb{Z}\,4947 \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{598+4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+598\mathbf{i}+\mathbf{ij}}{2}$
defines an isogeny $E_0 \to E_1$ of degree $4947 = 3 \cdot 17 \cdot 97$.

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Constructively, *partially* known endomorphism rings are useful.
$\rightsquigarrow$ (**Oriented curves** and) **isogeny group actions**. 🏝️

Signing with isogenies à la SQIsign

- Fiat–Shamir: signature scheme from identification scheme.

$$E_0 \xrightarrow{\quad secret \quad} E_{pk}$$

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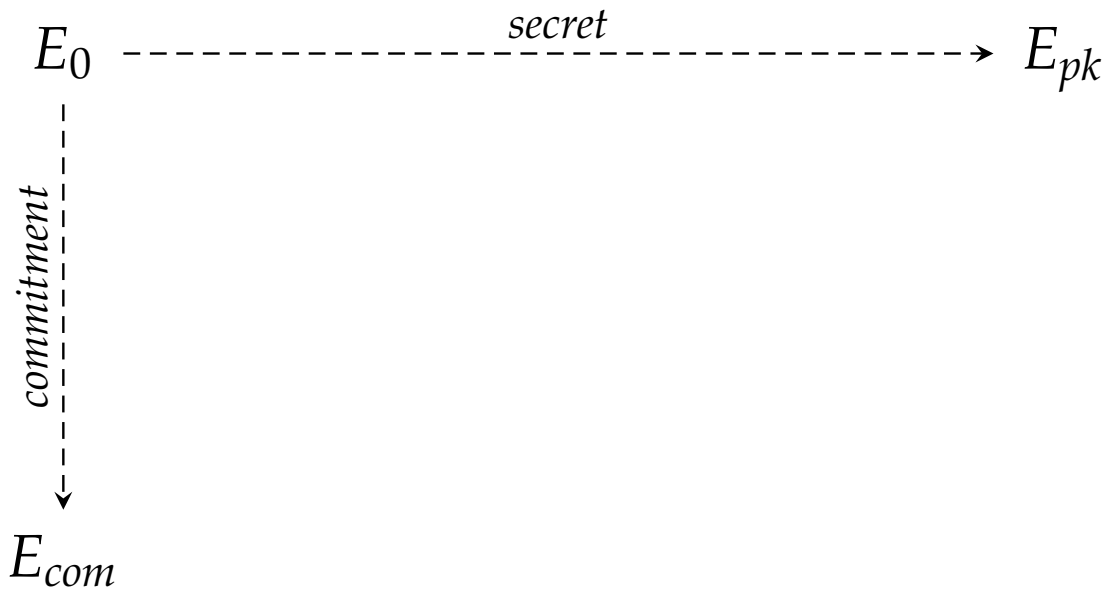

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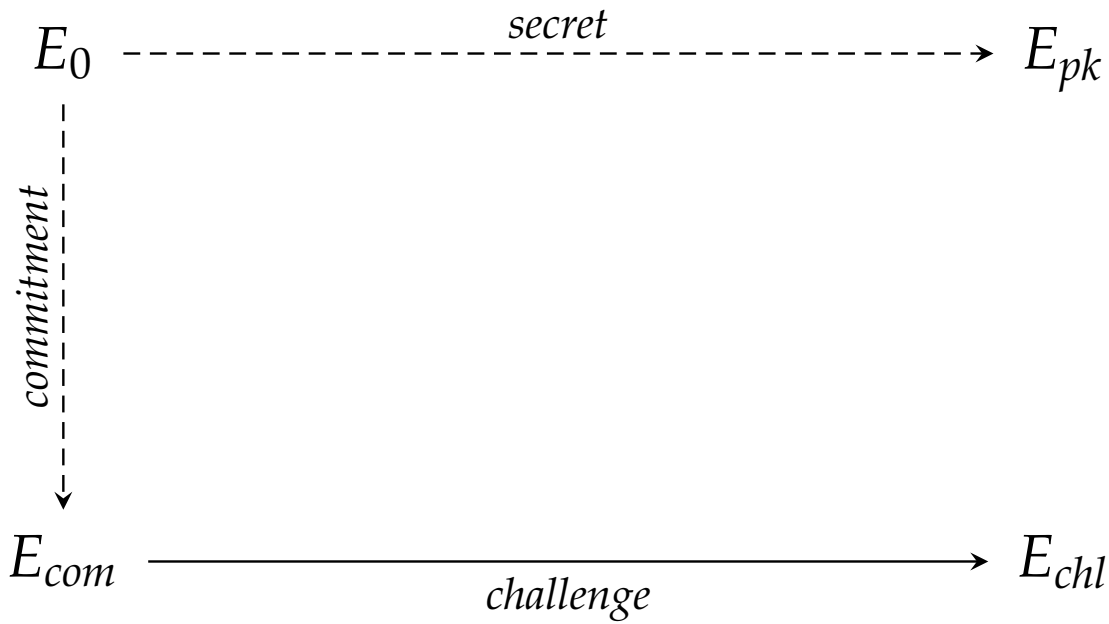

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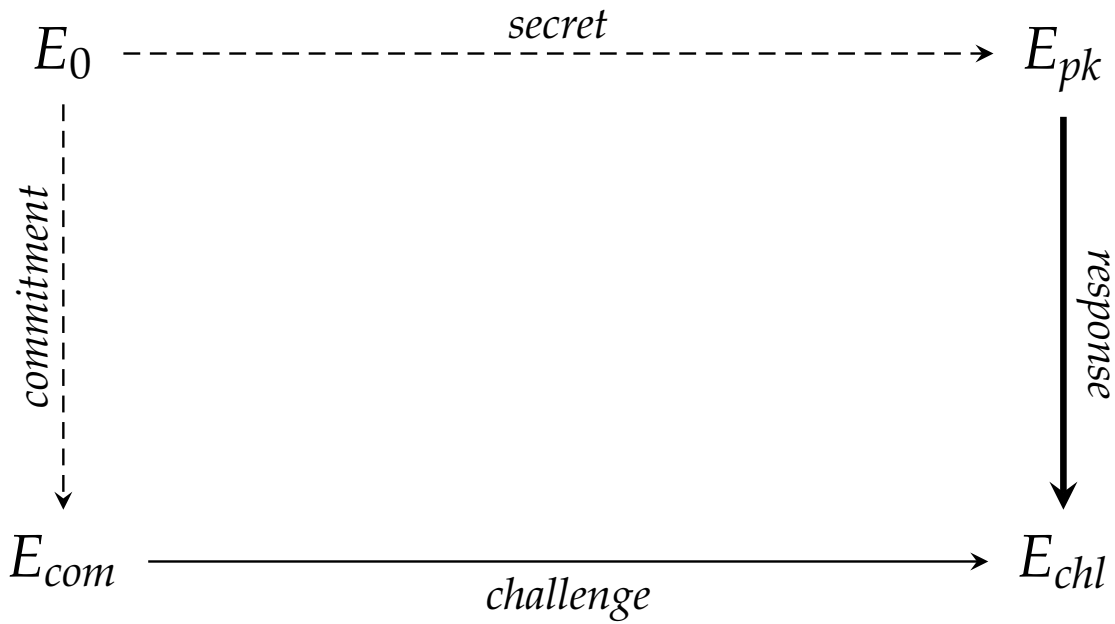

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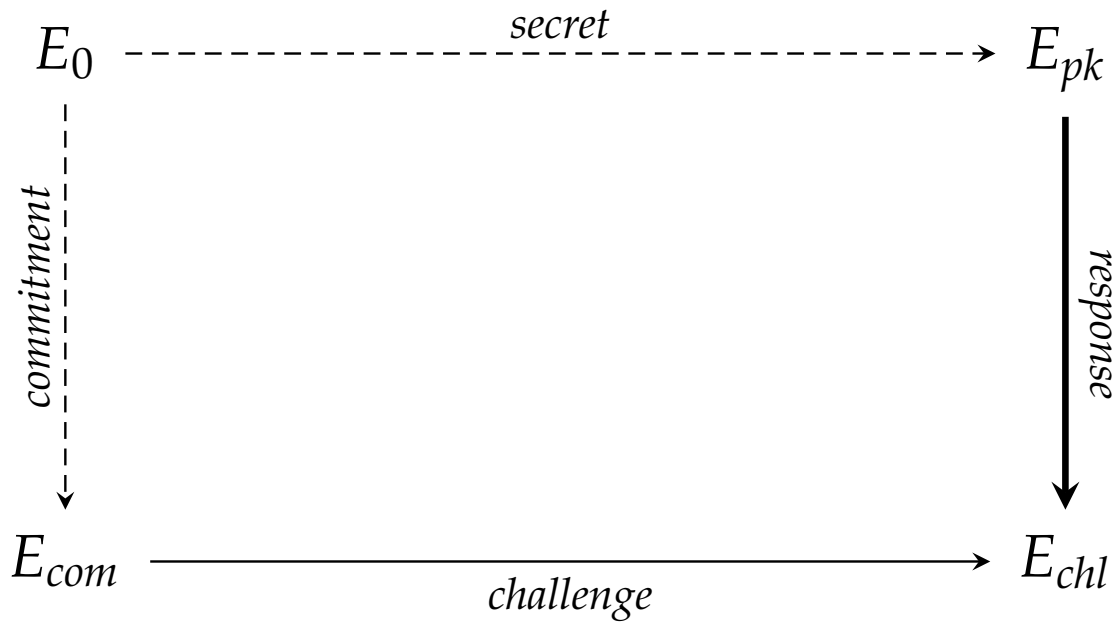

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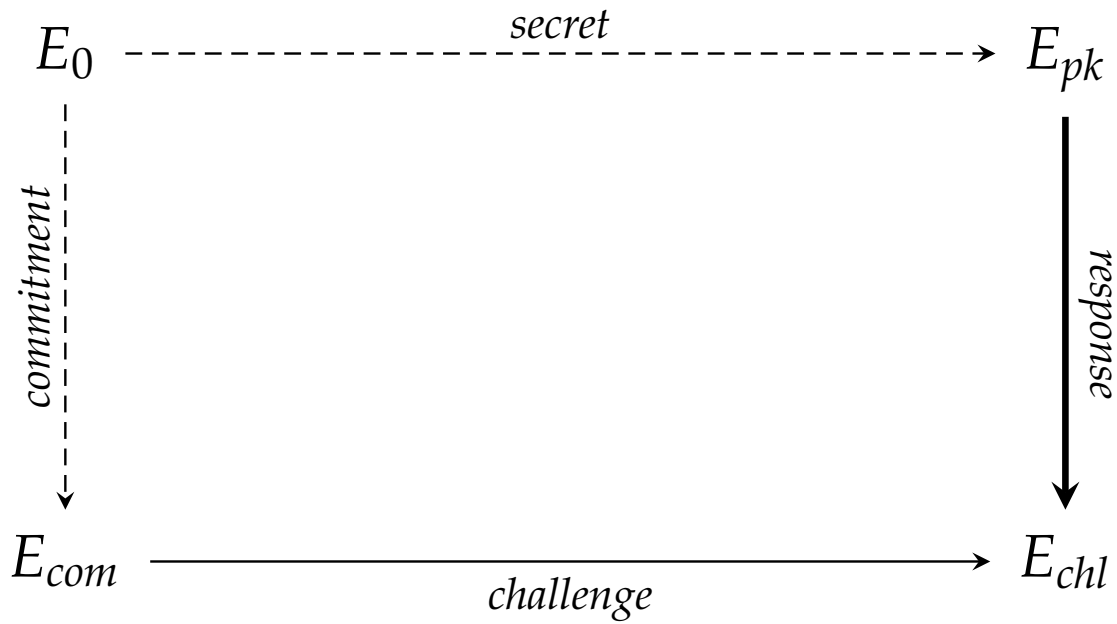

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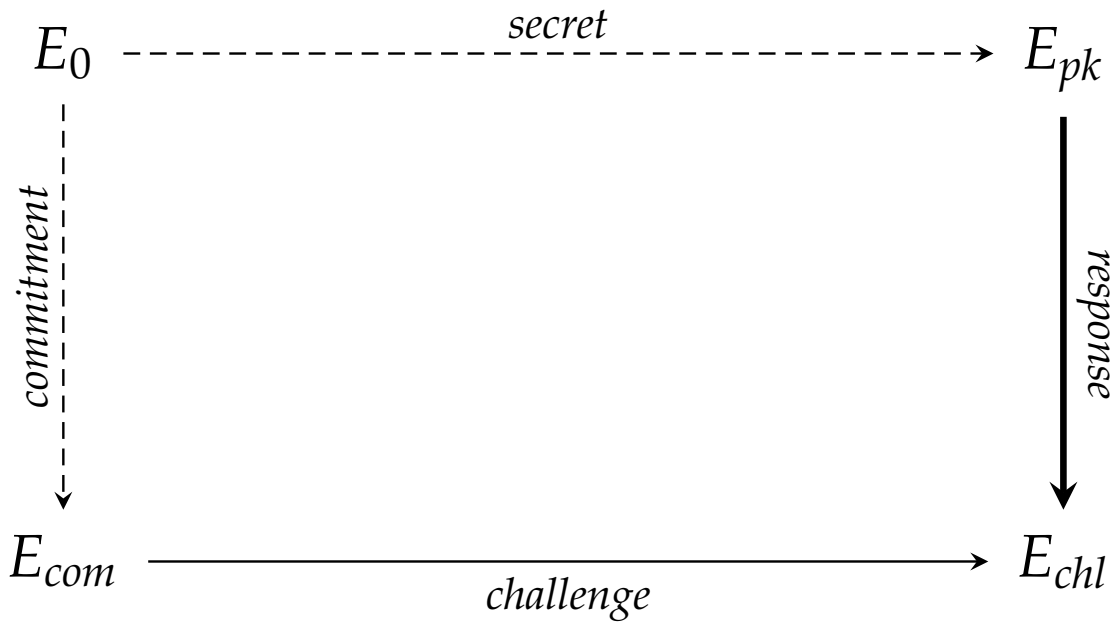
- **SQIsign**: Construct new path $E_{pk} \to E_{chl}$ (using *secret*).

Signing with isogenies à la SQIsign

- Fiat–Shamir: signature scheme from identification scheme.



- Easy signature: $E_{pk} \to E_0 \to E_{com} \to E_{chl}$. *Obviously broken.*
- **SQIsign**: Construct new path $E_{pk} \to E_{chl}$ (using *secret*).
- It relies on an explicit form of the Deuring correspondence.

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> *"If you have KLPT implemented very nicely as a black box, then anyone can implement SQIsign."*
> — Yan Bo Ti

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Known attacks for endomorphism-ring problem:
$O(\sqrt{p})$ classically, $O(\sqrt[4]{p})$ quantumly. Fully exponential! 🎉

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SQIsign (original version): Numbers

sizes

parameter set	public keys	signatures
NIST-**I**	**64** bytes	**177** bytes
NIST-**III**	**96** bytes	**263** bytes
NIST-**V**	**128** bytes	**335** bytes

performance

Cycle counts for a *generic C implementation* running on an Intel *Ice Lake* CPU.
Optimizations are certainly possible and work in progress.

parameter set	keygen	signing	verifying
NIST-**I**	**3728** megacycles	**5779** megacycles	**108** megacycles
NIST-**III**	**23734** megacycles	**43760** megacycles	**654** megacycles
NIST-**V**	**91049** megacycles	**158544** megacycles	**2177** megacycles

Source: `https://sqisign.org` (2023–2024)

☺ Timings have gotten *much* better since. $\rightsquigarrow$ *Friday.* ☺

Plan for this talk

- Some high-level intuition. ✓
- Elliptic curves & isogenies. ✓
- Isogeny group actions. ✓
- Signatures from isogenies. ✓

Ad break

https://isogeny.club

Questions?

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