# Ideal-to-isogeny algorithms: An overview

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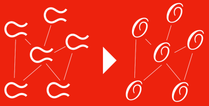
- C/R–S/DF–K–S/CSIDH/SCALLOP(-HD)/Clapoti(s)

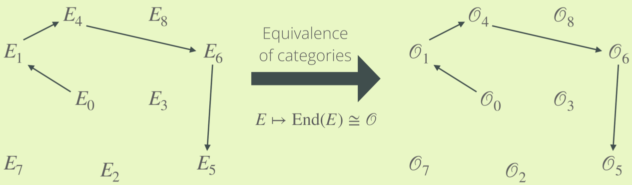
# Part 1: Deuring



# The Deuring Correspondence

## Deuring correspondence

**world of supersingular curves** | **world of maximal orders**




## curve-order dictionary

| **supersingular curves** | **quaternion orders** |
|---|---|
| curve $E$ (up to Galois conjugacy) | maximal order $\mathcal{O}$ (up to isomorphism) |
| isogeny $\varphi : E_1 \to E_2$ | integral ideal $I_\varphi$ that is left $\mathcal{O}_1$-ideal and right $\mathcal{O}_2$-ideal |
| endomorphism $\psi : E \to E$ | principal ideal $(\beta) \subset \mathcal{O}$ |
| and this continues for the *degree*, the *dual*, *equivalence*, *composition*... | and this continues for the *norm*, the *dual*, *equivalence*, *multiplication*... |

Radboud University

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- **2021**: Wesolowski assumes GRH and gives a provably polynomial-time variant.

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- Universe: Characteristic $p$. Assume $p \geq 5$.
- Supersingular elliptic curves: $E[p] = \{\infty\}$.
- Isogenies, endomorphisms, and so on and so forth.
- Famous examples:
  - $p \equiv 3 \pmod 4$ and $E\colon y^2 = x^3 + x$ with $j$-invariant 1728.
  - $p \equiv 2 \pmod 3$ and $E\colon y^2 = x^3 + 1$ with $j$-invariant 0.

# Computationally...

- We work with curves defined over $\mathbb{F}_{p^2}$ such that $\pi = [-p]$.
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  (This choice is natural: It includes the base-changes of curves defined over $\mathbb{F}_p$.)
- The group structure is known over all extensions:
  $E(\mathbb{F}_{p^{2k}}) \cong \mathbb{Z}/n \times \mathbb{Z}/n$ where $n = p^k - (-1)^k$.

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- Multiplication defined by relations $\mathbf{i}^2 = -q$, $\mathbf{j}^2 = -p$, $\mathbf{ji} = -\mathbf{ij}$.
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- The algebra $B_{p,\infty}$ has a conjugation $\overline{\phantom{x}}$ which negates $\mathbf{i}, \mathbf{j}, \mathbf{ij}$.
  The norm and trace of an element $\alpha$ are $\alpha\overline{\alpha} \in \mathbb{Z}_{\geq 0}$ and $\alpha + \overline{\alpha} \in \mathbb{Z}$.

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  We say $I$ connects $\mathcal{O}$ and $\mathcal{O}'$ if $\mathcal{O}I \subseteq I$ and $I\mathcal{O}' \subseteq I$.

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General theme: Things are easy in quaternion land.

# From curves to quaternions

$$E \mapsto \mathcal{O}$$

# Example #1

Assume $p \equiv 3 \pmod 4$.

Then $E\colon y^2 = x^3 + x$ is supersingular, and it has endomorphisms

$$\begin{aligned} \iota\colon \ (x,y) &\longmapsto (-x, \sqrt{-1}\cdot y)\,, \\ \pi\colon \ (x,y) &\longmapsto (x^p, y^p)\,. \end{aligned}$$

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In fact, the image in $B_{p,\infty}$ of a $\mathbb{Z}$-basis of $\mathrm{End}(E)$ is given by

$$\{1, \quad \mathbf{i}, \quad (\mathbf{i}+\mathbf{j})/2, \quad (1+\mathbf{ij})/2\}\,.$$

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**Lemma 10.** *Let $p$ be a prime number and $q, q' \in \mathbb{Z}_{>0}$ such that $B = (-q, -p \mid \mathbb{Q})$ and $B' = (-q', -p \mid \mathbb{Q})$ are quaternion algebras ramified at $p$ and $\infty$.*

*Then there exist $x, y \in \mathbb{Q}$ such that $x^2 + py^2 = q'/q$. Writing $1, \mathbf{i}', \mathbf{j}', \mathbf{k}'$ for the generators of $B'$ and $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ for the generators of $B$, and setting $\gamma := x + y\mathbf{j}$, the mapping*

$$\mathbf{i}' \mapsto \mathbf{i}\gamma, \qquad \mathbf{j}' \mapsto \mathbf{j}, \qquad \mathbf{k}' \mapsto \mathbf{k}\gamma$$

*defines a $\mathbb{Q}$-algebra isomorphism $B' \xrightarrow{\sim} B$.*

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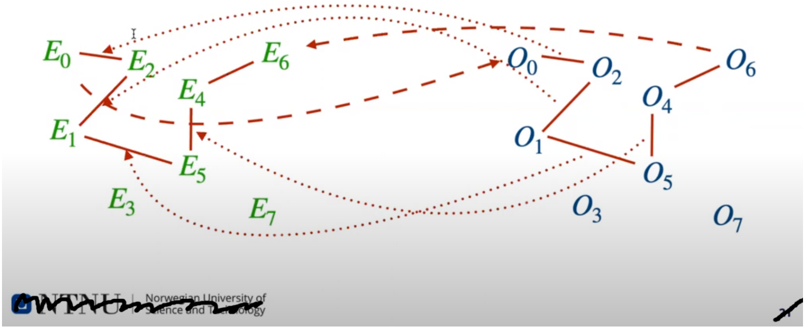

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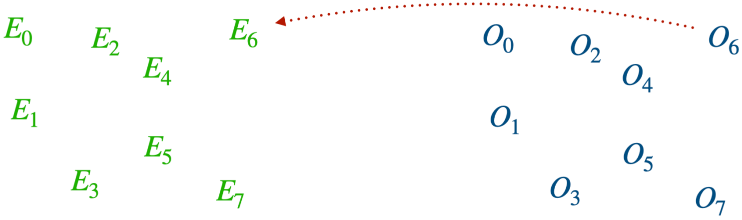

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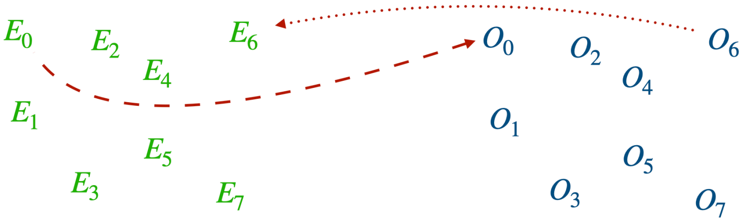


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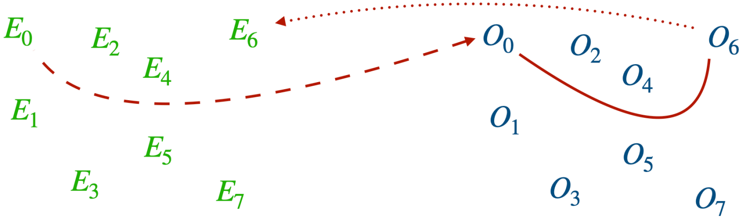


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  Solve the "isogeny problem" in quaternion land.

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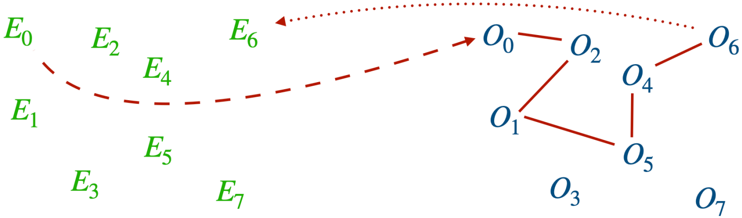


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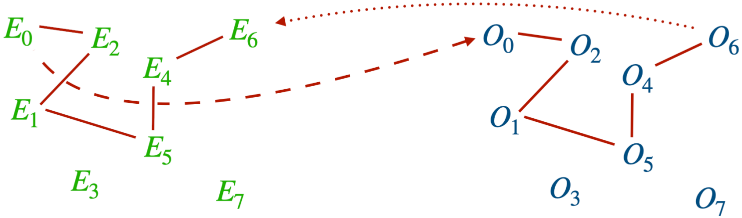


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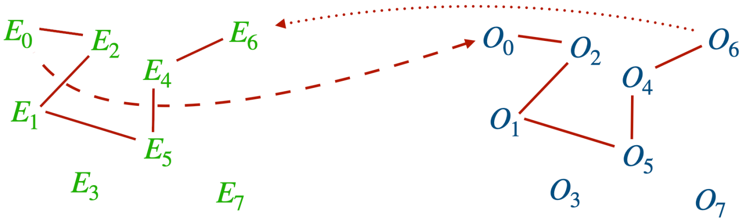
# From quaternions to curves

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I will talk about these *in reverse order*.

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Crucial observation: Complexity depends on factorization of $N$.
$\therefore$ No choice in $N$: It's the norm of $I$.

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...finds an equivalent ideal $J = I\overline{\gamma}/N$ of controlled norm $N'$.

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$\rightsquigarrow$ Do it twice with coprime degrees to evaluate on any point.

# Advertisement: *Deuring for the People!*

So we now know a way to do it, but how do we *actually* do it?

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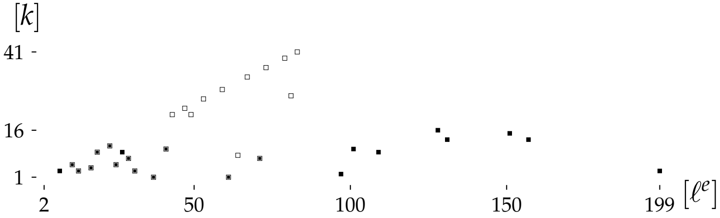
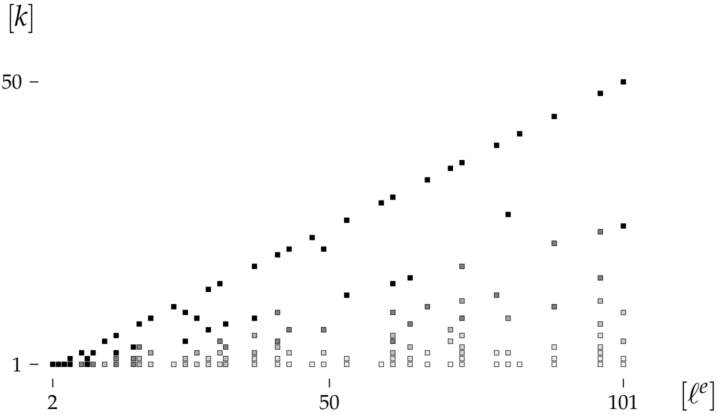


# Heatmap




Average extension $k$ required to access $\ell^e$-torsion.

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**Input:** Elliptic curve $E/\mathbb{F}_q$, minimal polynomial $f \in \mathbb{F}_q[X]$ of a subgroup $G \leq E$, isogeny $\varphi \colon E \to E'$ defined over $\mathbb{F}_q$.

**Output:** Minimal polynomial $f^\varphi \in \mathbb{F}_q[X]$ of the subgroup $\varphi(G) \leq E'$.

1 Write the x-coordinate map of $\varphi$ as a fraction $g_1/g_2$ of polynomials $g_1, g_2 \in \mathbb{F}_q[X]$.
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Complexity: $O(k^2) + \widetilde{O}(n)$. Naïvely $O(nk(\log k)^{O(1)})$.

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  Algorithm to compute a *normalized* degree-$q$ isogeny in time $\widetilde{O}(q)$.
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- Ingredient #3: Ibukiyama's theorem.
  Explicit basis for a maximal order of $B_{p,\infty}$ with an endomorphism $\sqrt{-q}$.
  In fact, such a maximal order is almost unique.

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  ?? Are we waiting for proper endomorphism-ring code?

# Connecting ideals

Finding **a** connecting $(\mathcal{O}, \mathcal{O}')$-ideal is straightforward:

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2. That's all, but typically the norm of $\mathcal{O}\mathcal{O}'$ is horrible.
   (Also, it's integral only in trivial cases $\rightsquigarrow$ scale by denominator in $\mathbb{Z}$.)

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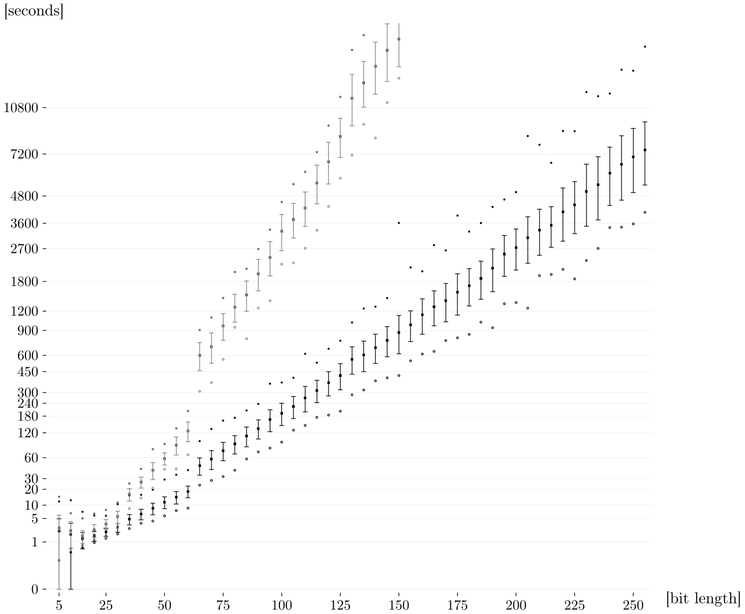
# Timings (SageMath, single core)



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We've been informed of one run for a 521-bit characteristic that took only about 7 hours.

$\rightsquigarrow$ Definitely practical for parameter setup etc.!

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∴ Algorithms for one "step" are quite technical.
See [ePrint 2022/234] and the more recent [ePrint 2023/1251].

# Part 2: The CM action

# The CM action on oriented curves

Now let $\mathcal{O}$ be an imaginary-quadratic order, say $\mathcal{O} = \mathbb{Z}[\vartheta]$.

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- Basic example: If $E/\mathbb{F}_q$ and $\pi \notin \mathbb{Z}$, then $\mathcal{O} = \mathbb{Z}[\pi]$ works.
- Other examples: $E/\mathbb{F}_{p^2}$ supersingular; many possible $\mathcal{O}$.

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- We consider $\mathcal{O}$-oriented elliptic curves: pairs $(E, \iota)$ with an explicit embedding $\iota \colon \mathcal{O} \to \mathrm{End}(E)$.
- Basic example: If $E/\mathbb{F}_q$ and $\pi \notin \mathbb{Z}$, then $\mathcal{O} = \mathbb{Z}[\pi]$ works.
- Other examples: $E/\mathbb{F}_{p^2}$ supersingular; many possible $\mathcal{O}$.

Ideals of $\mathcal{O}$ again define isogenies

$$\varphi \colon (E, \iota) \longrightarrow (E', \iota')$$

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$\Longrightarrow$ Compatibility for repeated applications of ideals of $\mathcal{O}$.
$\Longrightarrow$ Group action of $\mathrm{cl}(\mathcal{O})$ on such pairs!

# The basic strategy à la C/R–S

- Let $\mathfrak{l}_1, ..., \mathfrak{l}_n$ be small prime ideals of $\mathcal{O}$, and suppose $\mathfrak{a}$ is given to us in the form $\mathfrak{a} = \mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_n^{e_n}$.
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- Optimizations: Batch multiple $\mathfrak{l}_i$ together $\rightsquigarrow$ "strategies".

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<u>Issue:</u>

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  - $\rightsquigarrow$ Cost of evaluating after $k$ operations is $O(\exp(k))$.
- Representing $\mathrm{cl}(\mathcal{O})$ as reduced ideals allows computing in $\mathrm{cl}(\mathcal{O})$ efficiently, but evaluation becomes superpolynomial.

# Effective group actions à la CSI-FiSh/SCALLOP(-HD)

Partial solution:

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- To evaluate the action, solve a close(st)-vector problem.
  $\rightsquigarrow$ short equivalent exponent vector!

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- To evaluate the action, solve a close(st)-vector problem.
- CSI-FiSh: This is practically fast for CSIDH-512.
- Still, it’s asymptotically the bottleneck!

  https://yx7.cc/blah/2023-04-14.html

# SCALLO, PhD

# Even more maritime isogenies??

## Noun [ edit ]

**clapotis** *m* (*plural* **clapotis**)

1. lapping of water against a surface [synonyms ▲]

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- Kani: This gives an $N$-isogeny $F\colon\ E \times E \to E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}}$,

$$(P, Q) \mapsto (\phi_{\mathfrak{b}}(P) + \widehat{\psi_{\overline{\mathfrak{c}}}}(Q),\ -\phi_{\overline{\mathfrak{c}}}(P) + \widehat{\psi_{\mathfrak{b}}}(Q)).$$

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$$\ker(F) = \left\{([\mathrm{norm}(\mathfrak{b})]R, \gamma(R)) \mid R \in E[N]\right\}$$

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Let us explain the case of the specific isogeny $F$ to illustrate the usefulness of the module representation. We have $\mathfrak{b} = \frac{\overline{\gamma_b}}{\mathcal{N}(\mathfrak{a})}\mathfrak{a}$, so the multiplication map $\overline{\gamma_b}/\mathcal{N}(\mathfrak{a})\colon (\mathfrak{a}, \mathcal{N}(\cdot)/\mathcal{N}(\mathfrak{a})) \to (\mathfrak{b}, \mathcal{N}(\cdot)/\mathcal{N}(\mathfrak{b}))$ is an isomorphism $\alpha_b$ of unimodular Hermitian modules. The isogeny $\phi_{\mathfrak{b}} : E \to E_{\mathfrak{a}}$ corresponds from the module point of view to the post-composition of $\alpha_b$ with the natural $\mathcal{N}(\mathfrak{b})$-similitude given by the inclusion $(\mathfrak{b}, \mathcal{N}(\cdot)/\mathcal{N}(\mathfrak{b})) \to (R, \mathcal{N}(\cdot))$.

Likewise, the isogeny $F$ from Proposition 2.1 corresponds to a $N$-similitude $\psi\colon (\mathfrak{a}, \mathcal{N}(\cdot)/\mathcal{N}(\mathfrak{a})) \oplus (\overline{\mathfrak{a}}, \mathcal{N}(\cdot)/\mathcal{N}(\mathfrak{a})) \to (R, \mathcal{N}(\cdot)) \oplus (R, \mathcal{N}(\cdot))$.

The anti-equivalence of categories is exact, so the kernel of $F$ corresponds to the cokernel of $\psi$. Fix two generators of $\mathfrak{a}$, these generators induce surjective maps $R^2 \twoheadrightarrow \mathfrak{a}$, $R^2 \twoheadrightarrow \overline{\mathfrak{a}}$. Pre-composing $\psi$ with these epimorphisms, we get a module map $\tilde{\psi}\colon R^4 \to R^2$, whose cokernel is exactly the cokernel of $\psi$. The map $\tilde{\psi}$ is given by a $4 \times 2$ matrix of elements of $R$, hence of endomorphisms on $E$, and corresponds on the abelian variety side to a morphism $\tilde{\Phi}\colon E^2 \to E^4$. By exactness, the cokernel $\operatorname{coker}\tilde{\psi} = \operatorname{coker}\psi$, which as we have seen corresponds to $\operatorname{Ker} F$, is given by $\operatorname{Ker}\tilde{\Phi}$ which we can explicitly compute since the orientation by $R$ is effective on $E$.

# Polynomial-time group action: Clapoti(s)

- Minor detail: $N$ has no reason at all to be "nice".

$\rightsquigarrow$ $\{4, 8\}$-dimensional isogenies, per the usual...

# Interlude

# We could've had it all (5 years ago) [ePrint 2018/665]

$$(\mathfrak{a}_1 * E) \times \cdots \times (\mathfrak{a}_n * E) \cong (\mathfrak{a}_1 \cdots \mathfrak{a}_n) * E \times E^{n-1}. \tag{4}$$

and more generally, we have

$$\begin{aligned} (\mathfrak{a}_1 * E) \times \cdots \times (\mathfrak{a}_n * E) \cong (\mathfrak{a}'_1 * E) \times \cdots \times (\mathfrak{a}'_n * E) \quad & \text{if and only if} \\ \mathfrak{a}_1 \cdots \mathfrak{a}_n = \mathfrak{a}'_1 \cdots \mathfrak{a}'_n \quad & \text{as ideal classes in } \mathrm{Cl}(\mathscr{O}). \end{aligned} \tag{5}$$

As a side note, we now mention that those properties can *in part* be established using elementary techniques. More precisely, (4) is a consequence of the following elementary result.

**Theorem A.1.** *Let $E$ be an elliptic curve over a finite field $\mathbb{F}_q$, and $K$ a finite étale subgroup of $E$ (i.e., the map $E \to E/K$ is separable) defined over $\mathbb{F}_q$. Suppose that $K$ contains subgroups $K_i$ defined over $\mathbb{F}_q$, for $1 \leq i \leq n$, whose orders are pairwise coprime, and suppose $K = K_1 + \cdots + K_n$. Then:*

$$(E/K_1) \times \cdots \times (E/K_n) \cong (E/K) \times E^{n-1}.$$

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$$(E/K_1) \times \cdots \times (E/K_n) \cong (E/K) \times E^{n-1}.$$

*Proof.* The result is immediate for $n = 1$. We next prove the result for $n = 2$ by constructing an explicit isomorphism. Consider the commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, "\theta"] & E/K_1 \arrow[d, "\psi_1"] \\ E/K_2 \arrow[r, "\psi_2"'] & E/K \end{tikzcd}$$

where all maps are the natural quotient isogenies. If we denote by $m_1$ and $m_2$ the orders of $K_1$ and $K_2$, we have $\deg \varphi_1 = \deg \psi_2 = m_1$ and $\deg \varphi_2 = \deg \psi_1 = m_2$. Now choose integers $a, b \in \mathbb{Z}$ such that $am_1 + bm_2 = 1$. We define morphisms

$$f\colon E \times (E/K) \to (E/K_1) \times (E/K_2) \qquad \text{and} \qquad g\colon (E/K_1) \times (E/K_2) \to E \times (E/K)$$

by the following matrices:

$$\mathrm{Mat}(f) = \begin{pmatrix} \varphi_1 & \widehat{\psi_1} \\ -b\varphi_2 & a\widehat{\psi_2} \end{pmatrix} \qquad \text{and} \qquad \mathrm{Mat}(g) = \begin{pmatrix} a\widehat{\varphi_1} & -\widehat{\varphi_2} \\ b\psi_1 & \psi_2 \end{pmatrix}.$$

Questions?