Deuring for the People:

Supersingular Elliptic Curves with Prescribed Endomorphism Ring in General Characteristic

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This talk: **How?**

(The $\Longleftarrow$ direction is exponential-time as far as we know.)

$\longrightarrow$ See for instance Annamaria Iezzi's talk in MS28 on Tuesday.

PSA

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- $\approx$All isogeny assumptions reduce to the $\Longleftarrow$ direction.
- **SQIsign** builds on the $\Longrightarrow$ direction constructively.
- Essential tool for both constructions and attacks.

History lesson

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Wenn aber $\mathfrak{R}$ eine vorgegebene Maximalordnung in $Q_{\infty,p}$ ist, in der der Primteiler von p Hauptideal ist, so gibt es genau eine Invariante j, zu der dieser Multiplikatorenring gehört, sie ist absolut rational. Ist der Primteiler von p kein Hauptideal, so gibt es zwei konjugierte Invarianten vom Absolutgrad 2 zu diesem Multiplikatorenring. Die Anzahl der j, zu denen eine Maximalordnung von $Q_{\infty,p}$ als Multiplikatorenring gehört, ist gleich der Klassenzahl von $Q_{\infty,p}$.

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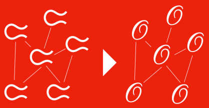
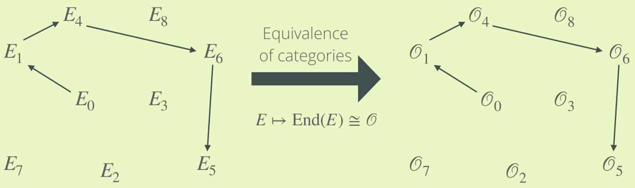
- **2021**: Wesolowski assumes GRH and gives a provably polynomial-time variant.



The Deuring Correspondence

Deuring correspondence

world of supersingular curves / **world of maximal orders**



curve-order dictionary

supersingular curves	**quaternion orders**
curve E (up to Galois conjugacy)	maximal order $\mathcal{O}$ (up to isomorphism)
isogeny $\varphi : E_1 \to E_2$	integral ideal I_φ that is left $\mathcal{O}_1$-ideal and right $\mathcal{O}_2$-ideal
endomorphism $\psi : E \to E$	principal ideal $(\beta) \subset \mathcal{O}$
and this continues for the *degree*, the *dual*, *equivalence*, *composition*...	and this continues for the *norm*, the *dual*, *equivalence*, *multiplication*...

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- Supersingular elliptic curves: $E[p] = \{\infty\}$.
- Isogenies, endomorphisms, and so on and so forth.
- Famous examples:
  - $p \equiv 3 \pmod 4$ and $E\colon y^2 = x^3 + x$ with j-invariant 1728.
  - $p \equiv 2 \pmod 3$ and $E\colon y^2 = x^3 + 1$ with j-invariant 0.

Computationally...

- We work with curves defined over $\mathbb{F}_{p^2}$ such that $\pi = [-p]$.
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  (This choice is natural: It includes the base-changes of curves defined over $\mathbb{F}_p$.)
- The group structure is known over all extensions:
  $E(\mathbb{F}_{p^{2k}}) \cong \mathbb{Z}/n \times \mathbb{Z}/n$ where $n = p^k - (-1)^k$.

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- Multiplication defined by relations $\mathbf{i}^2 = -q$, $\mathbf{j}^2 = -p$, $\mathbf{ji} = -\mathbf{ij}$.
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  ⚠ All valid q define isomorphic algebras $B_{p,\infty}$.
- The algebra $B_{p,\infty}$ has a conjugation $\overline{}$ which negates $\mathbf{i}, \mathbf{j}, \mathbf{ij}$.
  The norm and trace of an element α are $\alpha\overline{\alpha} \in \mathbb{Z}_{\geq 0}$ and $\alpha + \overline{\alpha} \in \mathbb{Z}$.

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  We say I connects $\mathcal{O}$ and $\mathcal{O}'$ if $\mathcal{O}I \subseteq I$ and $I\mathcal{O}' \subseteq I$.

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General theme: Things are easy in quaternion land.

From curves to quaternions

$$E \mapsto \mathrm{End}(E)$$

Example #1

Assume $p \equiv 3 \pmod 4$.

Then $E\colon y^2 = x^3 + x$ is supersingular, and it has endomorphisms

$$\begin{aligned} \iota\colon\ (x,y) &\longmapsto (-x, \sqrt{-1}\cdot y)\,, \\ \pi\colon\ (x,y) &\longmapsto (x^p, y^p)\,. \end{aligned}$$

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In fact, the image in $B_{p,\infty}$ of a $\mathbb{Z}$-basis of $\mathrm{End}(E)$ is given by

$$\{1, \quad \mathbf{i}, \quad (\mathbf{i}+\mathbf{j})/2, \quad (1+\mathbf{ij})/2\}\,.$$

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  - Multiple q define the *same* $B_{p,\infty}$.
    Need to convert from $\mathbf{i}^2 = -q$ basis to $\mathbf{i}'^2 = -q'$ basis.

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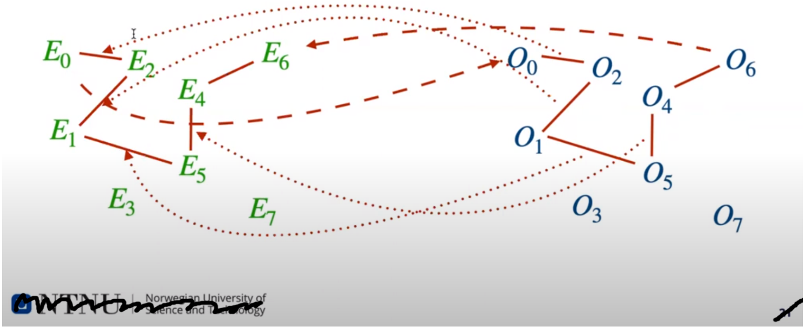

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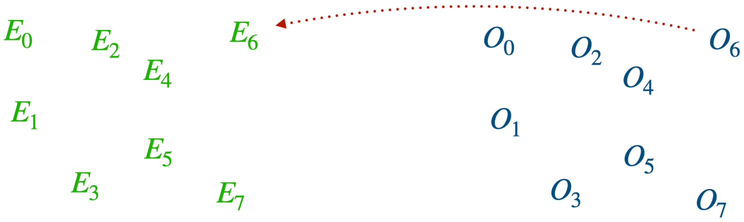

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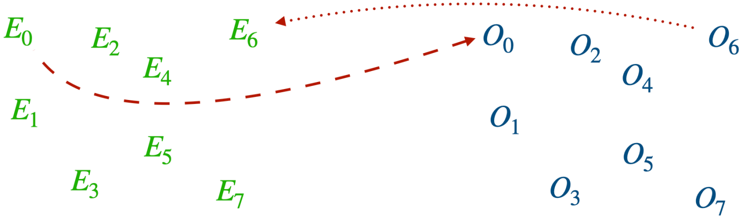

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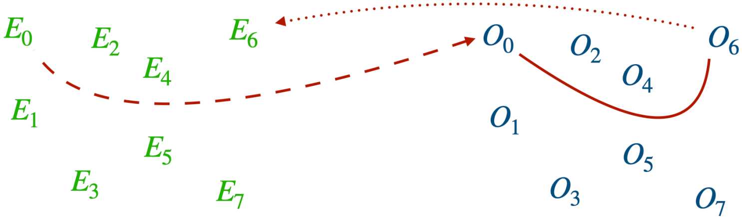

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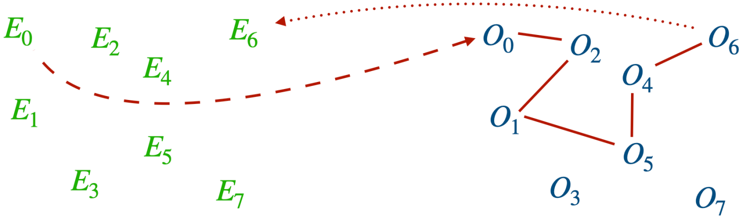

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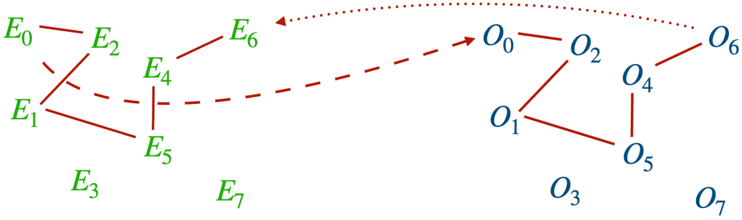

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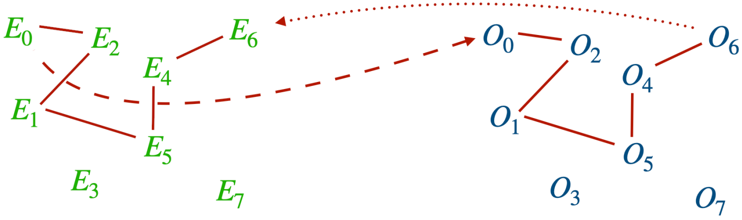

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I will talk about these *in reverse order*.

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Crucial observation: Complexity depends on factorization of N.

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  Then $H_I = \langle H_1', ..., H_r' \rangle$.
- If φ_I is cyclic, we have $\ker(\alpha|_{E[N]}) = \overline{\alpha}(E[N])$. No logarithms!

Crucial observation: Complexity depends on factorization of N.
$\frown$ No choice in N: It's the norm of I.

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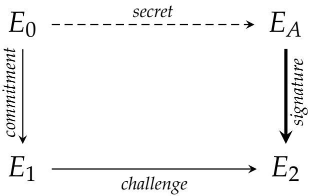
Fact: Equivalent ideals $\rightsquigarrow$ isomorphic codomains.

SQIsign

...is a signature scheme based on the Deuring correspondence.



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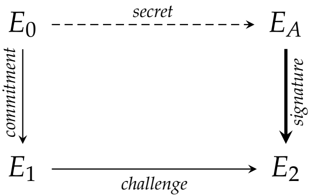

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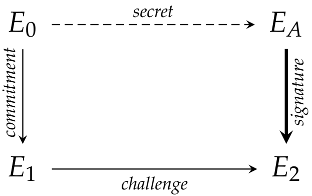

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- Cryptographic reductions and general computer algebra want it to be fast for arbitrary fields. $\rightsquigarrow$ Our work!

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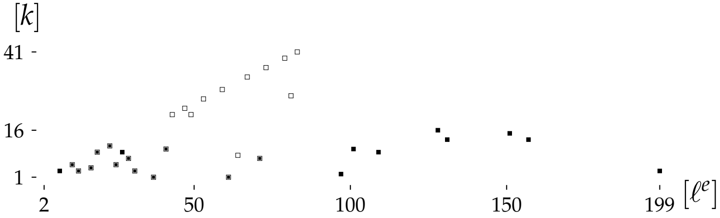
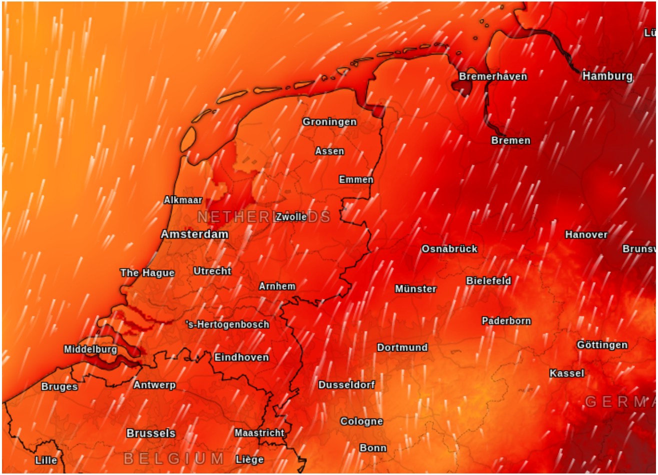


Heatmap



Bremerhaven
Hamburg
Groningen
Assen
Bremen
Emmen
Alkmaar
NETHERLANDS
Zwolle
Amsterdam
Hanover
Osnabrück
The Hague
Utrecht
Arnhem
Münster
Bielefeld
's-Hertogenbosch
Paderborn
Middelburg
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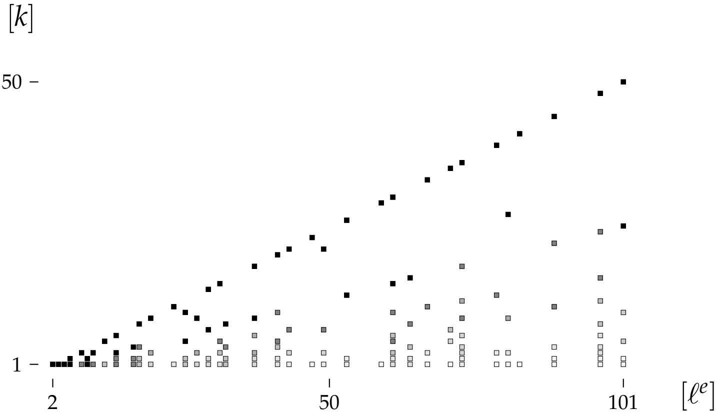

Average extension k required to access ℓ^e-torsion.

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Algorithm 5: `PushSubgroup`(E, f, φ)

Input: Elliptic curve $E/\mathbb{F}_q$, minimal polynomial $f \in \mathbb{F}_q[X]$ of a subgroup $G \leq E$, isogeny $\varphi\colon E \to E'$ defined over $\mathbb{F}_q$.

Output: Minimal polynomial $f^\varphi \in \mathbb{F}_q[X]$ of the subgroup $\varphi(G) \leq E'$.

Write the x-coordinate map of φ as a fraction g_1/g_2 of polynomials $g_1, g_2 \in \mathbb{F}_q[X]$.
Let $g_{\text{ker}} \leftarrow \gcd(g_2, f)$ and $f_1 \leftarrow f/g_{\text{ker}}$.
Compute $g_1 \cdot g_2^{-1} \bmod f_1 \in \mathbb{F}_q[X]$ and reinterpret it as a quotient-ring element $\alpha \in \mathbb{F}_q[X]/f_1$.
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Complexity: $O(k^2) + \widetilde{O}(n)$. Naïvely $O(nk(\log k)^{O(1)})$.

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- Ingredient #3: Ibukiyama's theorem.
  Explicit basis for a maximal order of $B_{p,\infty}$ with an endomorphism $\sqrt{-q}$.
  In fact, such a maximal order is almost unique.

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    + 16579/2*k, -2147483647 - 9708*j + 12777*k, -2147483647
    - 2147483647*i - 22485*j + 3069*k)
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sage: E1, phi, _ = constructive_deuring(I, E0, iota)
sage: phi
Composite morphism of degree 14763897348161206530374369280
          = 2^29*3^3*5*7^2*11*13*17*31*41*43^2*61*79*151:
  From: Elliptic Curve defined by y^2 = x^3 + x over
          Finite Field in i of size 2147483647^2
  To:   Elliptic Curve defined by y^2 = x^3 + (1474953432*i
              +1816867654)*x + (581679615*i+260136654)
          over Finite Field in i of size 2147483647^2
```

Timings (SageMath, single core)