Computing the Deuring correspondence and applications to cryptography

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:-) The "$\Leftarrow$" direction is easy, the "$\Rightarrow$" direction seems hard!

$\rightsquigarrow$ *Cryptography!*

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- Grim future: Quantum computers are expected to break almost all of the systems we currently use.
- Solution: Post-quantum cryptography.
  It is based on different types of computational problems, including isogeny problems!

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Constructively, *partially* known endomorphism rings are useful.
$\rightsquigarrow$ **Oriented curves** and **the isogeny class-group action**.

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Theorem. The (contravariant) functor

$$E \longmapsto \mathrm{Hom}(E, E_0)$$

defines an equivalence of categories between

- supersingular elliptic curves with isogenies; and
- invertible left $\mathcal{O}_0$-modules with nonzero left $\mathcal{O}_0$-module homomorphisms.

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Corollary (Deuring). Isomorphism classes of supersingular elliptic curves are in bijection with the (left) class set $\mathrm{Cls}_L(\mathcal{O}_0)$.

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Important consequence: The isogeny $\varphi_I\colon E_0 \to E$ defined by a left $\mathcal{O}_0$-ideal I has kernel $\bigcap_{\alpha\in I} \ker\alpha \leq E_0$.

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- Moreover, then $\mathrm{End}(E) \hookrightarrow B_{p,\infty}$ via $\alpha \mapsto \widehat{\varphi}_I\alpha\varphi_I/\deg(\varphi_I)$.
- Under this embedding, $\mathrm{End}(E) = \{\alpha \in B_{p,\infty} : I\alpha \subseteq I\}$.

History and algorithms

- **1941**: Deuring proves the correspondence.

Wenn aber $\mathfrak{R}$ eine vorgegebene Maximalordnung in $Q_{\infty,p}$ ist, in der der Primteiler von p Hauptideal ist, so gibt es genau eine Invariante j, zu der dieser Multiplikatorenring gehört, sie ist absolut rational. Ist der Primteiler von p kein Hauptideal, so gibt es zwei konjugierte Invarianten vom Absolutgrad 2 zu diesem Multiplikatorenring. Die Anzahl der j, zu denen eine Maximalordnung von $Q_{\infty,p}$ als Multiplikatorenring gehört, ist gleich der Klassenzahl von $Q_{\infty,p}$.

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- **2023**: Eriksen–Panny–Sotáková–Veroni develop practical optimizations and publish a fully general implementation.

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- Isogenies, endomorphisms, and so on and so forth.
- Famous examples:
  - $p \equiv 3 \pmod 4$ and $E\colon y^2 = x^3 + x$ with j-invariant 1728.
  - $p \equiv 2 \pmod 3$ and $E\colon y^2 = x^3 + 1$ with j-invariant 0.

Computationally...

- We work with curves defined over $\mathbb{F}_{p^2}$ such that $\pi = [-p]$.
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- We construct isogenies from their kernel subgroups.
- We work with smooth-degree isogenies since classical isogeny formulas require exponential time in $\log(\mathit{degree})$.

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- Multiplication defined by relations $\mathbf{i}^2 = -q$, $\mathbf{j}^2 = -p$, $\mathbf{ji} = -\mathbf{ij}$.
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- The algebra $B_{p,\infty}$ has a conjugation $\overline{}$ which negates $\mathbf{i}, \mathbf{j}, \mathbf{ij}$.
  The norm and trace of an element α are $\alpha\overline{\alpha} \in \mathbb{Z}_{\geq 0}$ and $\alpha + \overline{\alpha} \in \mathbb{Z}$.

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  We say I connects $\mathcal{O}$ and $\mathcal{O}'$ if $\mathcal{O}I \subseteq I$ and $I\mathcal{O}' \subseteq I$.

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General theme: Things are easy in quaternion land.

From curves to quaternions

$$E \mapsto \mathcal{O}$$

Example #1

Assume $p \equiv 3 \pmod 4$.

Then $E\colon y^2 = x^3 + x$ is supersingular, and it has endomorphisms

$$\begin{aligned} \iota\colon \ (x,y) &\longmapsto (-x, \sqrt{-1}\cdot y)\,, \\ \pi\colon \ (x,y) &\longmapsto (x^p, y^p)\,. \end{aligned}$$

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In fact, the image in $B_{p,\infty}$ of a $\mathbb{Z}$-basis of $\mathrm{End}(E)$ is given by

$$\{1, \quad \mathbf{i}, \quad (\mathbf{i}+\mathbf{j})/2, \quad (1+\mathbf{ij})/2\}\,.$$

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As far as we know, these are *hard* problems (even quantumly).

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    Need to convert from $\mathbf{i}^2 = -q$ basis to $\mathbf{i}'^2 = -q'$ basis.

Lemma 10. *Let p be a prime number and $q, q' \in \mathbb{Z}_{>0}$ such that $B = (-q, -p \mid \mathbb{Q})$ and $B' = (-q', -p \mid \mathbb{Q})$ are quaternion algebras ramified at p and ∞.*

Then there exist $x, y \in \mathbb{Q}$ such that $x^2 + py^2 = q'/q$. Writing $1, \mathbf{i}', \mathbf{j}', \mathbf{k}'$ for the generators of B' and $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ for the generators of B, and setting $\gamma := x + y\mathbf{j}$, the mapping

$$\mathbf{i}' \mapsto \mathbf{i}\gamma, \qquad \mathbf{j}' \mapsto \mathbf{j}, \qquad \mathbf{k}' \mapsto \mathbf{k}\gamma$$

defines a $\mathbb{Q}$-algebra isomorphism $B' \xrightarrow{\sim} B$.

From quaternions to curves

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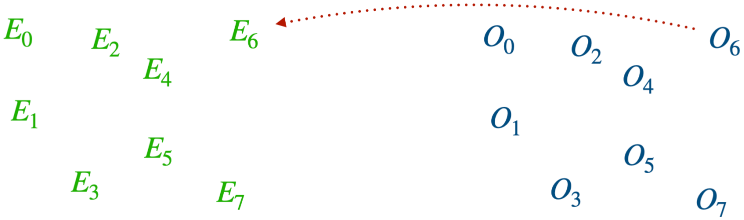

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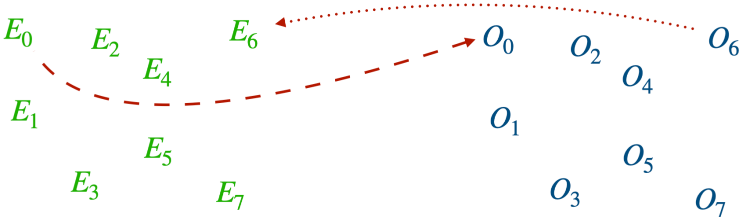

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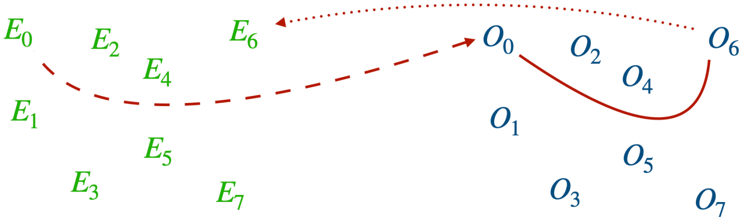

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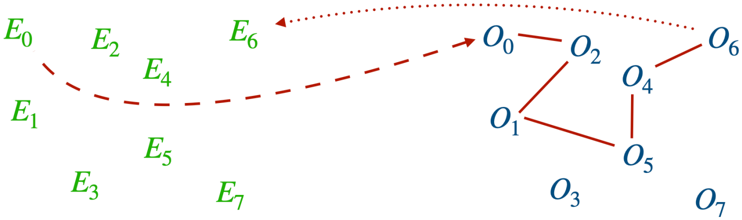

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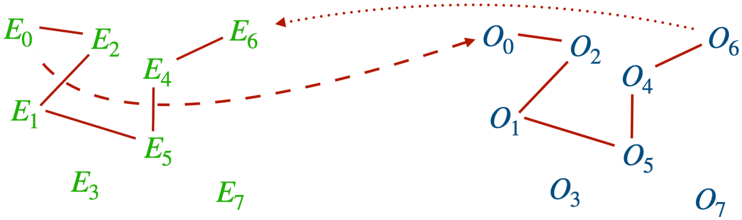

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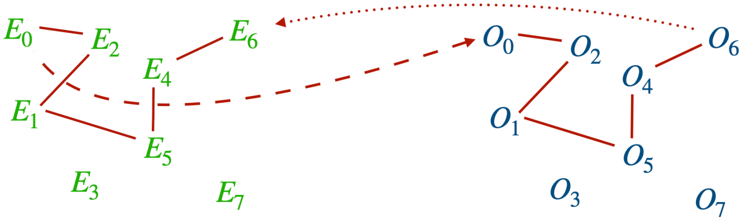

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I will talk about these *in reverse order*.

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Crucial observation: Complexity depends on factorization of N.

Step $0.\overline{9}$: Connecting ideals

Finding **a** connecting $(\mathcal{O}, \mathcal{O}')$-ideal is straightforward:

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2. That's all, but typically the norm of $\mathcal{O}\mathcal{O}'$ is horrible.
   (Also, it's integral only in trivial cases $\rightsquigarrow$ scale by denominator in $\mathbb{Z}$.)

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$\rightsquigarrow$ Do it twice with coprime degrees to evaluate on any point.

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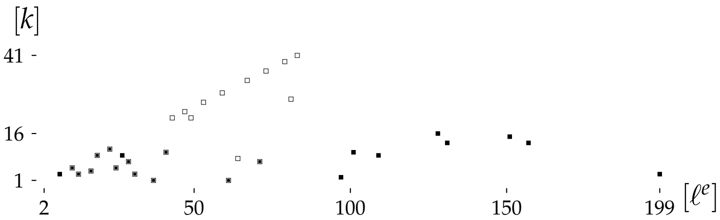
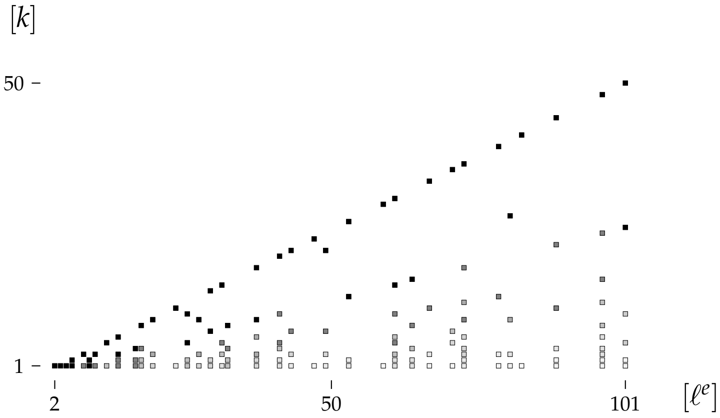


Heatmap



Average extension k required to access ℓ^e-torsion.

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- Ingredient #3: Ibukiyama's theorem.
  Explicit basis for a maximal order of $B_{p,\infty}$ with an endomorphism $\sqrt{-q}$.
  In fact, there are only very few maximal orders containing $\sqrt{-q}$.

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  To:   Elliptic Curve defined by y^2 = x^3 + (1474953432*i
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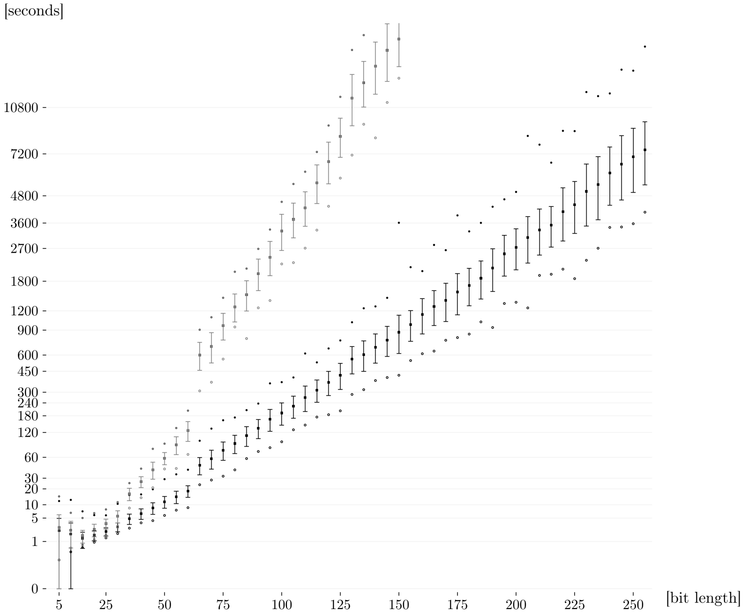
Timings (SageMath, single core)



Timings (SageMath, single core)

We've been informed of one run for a 521-bit characteristic that took only about 7 hours.

$\rightsquigarrow$ Definitely practical for parameter setup etc.!

SQIsign: What?

https://sqisign.org

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- A new and very hot post-quantum signature scheme.
- Part of NIST's post-quantum standardization process.

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$$E_0 \xrightarrow{\textit{secret}} E_A$$

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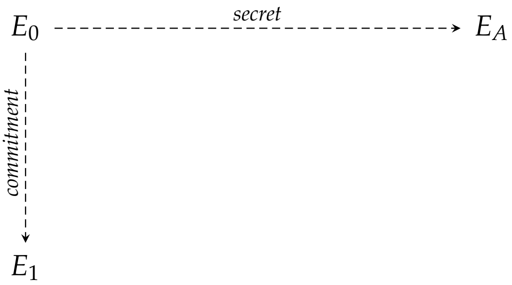

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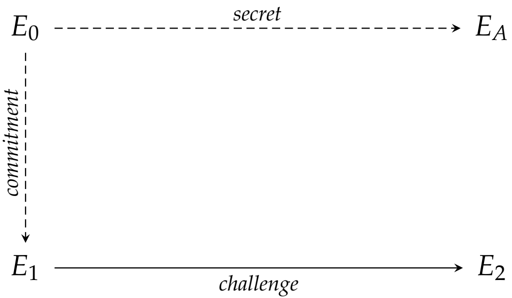

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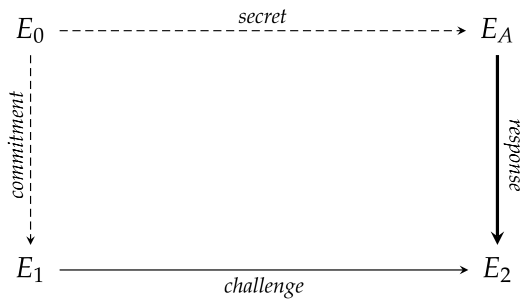

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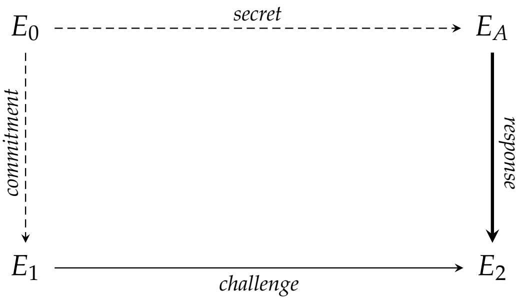

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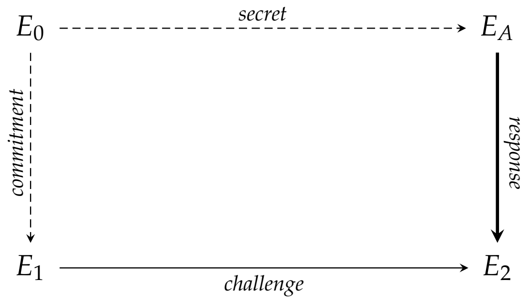

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- **SQIsign**'s solution: Construct new path $E_A \to E_2$ (using *secret*).

SQIsign: How?

Main idea:

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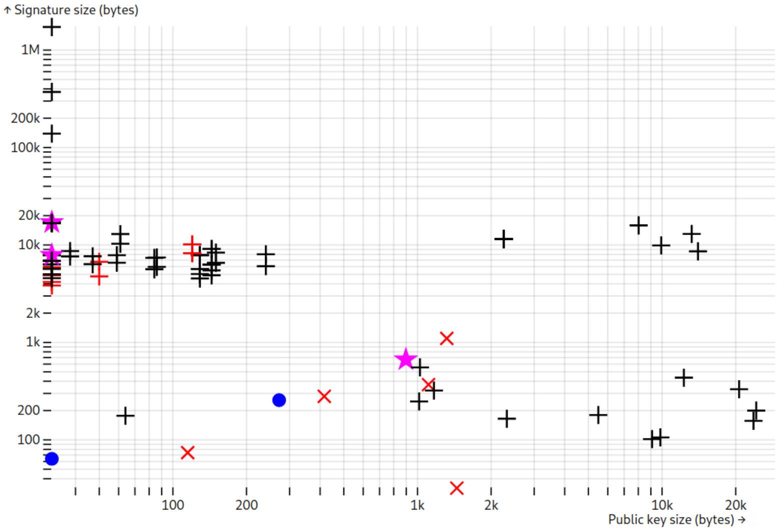
> *"If you have KLPT implemented very nicely as a black box, then anyone can implement SQIsign."* — Yan Bo Ti

SQIsign: Comparison



Source: `https://pqshield.github.io/nist-sigs-zoo`

Bonus slides

Gluing elliptic curves

Awesome new technique (established 2022):

Computing isogenies between *products* of elliptic curves

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  - Similar group structure, but more components.
  - Can define isogenies from kernel subgroups.

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- Similar to elliptic curves in many ways:
  - Points form an abelian group.
  - Similar group structure, but more components.
  - Can define isogenies from kernel subgroups.
- Computing with surfaces explicitly is possible, but painful. Everyone works with Jacobians of genus-2 curves instead.

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$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi'} \\ E'' & \xrightarrow[\varphi']{} & E''' \end{array}$$

where $a := \deg \varphi$ and $b := \deg \psi$ are coprime; let $N := a + b$.

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Lemma. Then

$$F := \begin{pmatrix} \varphi & \widehat{\psi'} \\ -\psi & \widehat{\varphi'} \end{pmatrix}$$

defines an N-isogeny $E \times E''' \to E' \times E''$.

Its kernel is $\ker(F) = \left\{ (\widehat{\varphi}(P), \psi'(P)) \mid P \in E'[N] \right\}$.

Representing $\varphi|_{E[N]}$

Recall: For embedding lemma, need to evaluate φ on $E[N]$.

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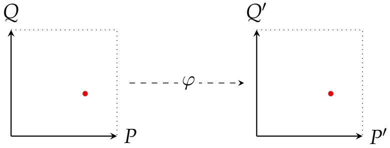

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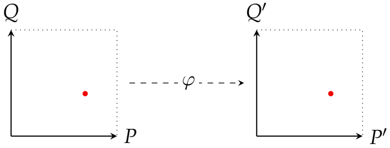

<u>Evaluating φ at an arbitrary point $T \in E[N]$:</u>

1. Decompose $T = [u]P + [v]Q$ with $u, v \in \mathbb{Z}$.
   This is a discrete-logarithm computation, which is easy whenever N is smooth!
2. Output $[u]P' + [v]Q'$.

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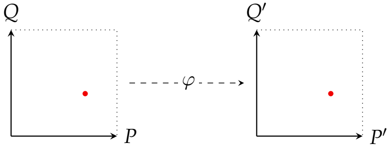

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$\Longrightarrow$ The data (P, Q, P', Q') encodes the *restriction* $\varphi|_{E[N]}$.

Questions?

(Also feel free to email me: `lorenz@yx7.cc`)