To the **End** and back

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Mathematics for post-quantum cryptanalysis

Budapest, 8 August 2024

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$\rightsquigarrow$ *Cryptography!*

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Constructively, *partially* known endomorphism rings are useful.
$\rightsquigarrow$ **Oriented curves** and **the isogeny class-group action**. 🏝️

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- Supersingular elliptic curves: $E[p] = \{\infty\}$.
- Isogenies, endomorphisms, and so on and so forth.
- Famous examples:
  - $p \equiv 3 \pmod 4$ and $E\colon y^2 = x^3 + x$ with j-invariant 1728.
  - $p \equiv 2 \pmod 3$ and $E\colon y^2 = x^3 + 1$ with j-invariant 0.

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- We work with curves defined over $\mathbb{F}_{p^2}$ such that $\pi = [-p]$.
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  (This choice is natural: It includes the base-changes of curves defined over $\mathbb{F}_p$.)
- The group structure is known over all extensions:
  $E(\mathbb{F}_{p^{2k}}) \cong \mathbb{Z}/n \times \mathbb{Z}/n$ where $n = p^k - (-1)^k$.

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- Multiplication defined by relations $\mathbf{i}^2 = -q$, $\mathbf{j}^2 = -p$, $\mathbf{ji} = -\mathbf{ij}$.
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  ⚠ All valid q define isomorphic algebras $B_{p,\infty}$.
- The algebra $B_{p,\infty}$ has a conjugation $\overline{}$ which negates $\mathbf{i}, \mathbf{j}, \mathbf{ij}$.
  The norm and trace of an element α are $\alpha\overline{\alpha} \in \mathbb{Z}_{\geq 0}$ and $\alpha + \overline{\alpha} \in \mathbb{Z}$.

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  We say I connects $\mathcal{O}$ and $\mathcal{O}'$ if $\mathcal{O}I \subseteq I$ and $I\mathcal{O}' \subseteq I$.

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General theme: Things are easy in quaternion land.

From curves to quaternions

$$E \mapsto \mathcal{O}$$

(The presumably hard direction.)

Example #1

Assume $p \equiv 3 \pmod 4$.

Then $E\colon y^2 = x^3 + x$ is supersingular, and it has endomorphisms

$$\begin{aligned} \iota\colon\ (x,y) &\longmapsto (-x, \sqrt{-1}\cdot y)\,, \\ \pi\colon\ (x,y) &\longmapsto (x^p, y^p)\,. \end{aligned}$$

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In fact, the image in $B_{p,\infty}$ of a $\mathbb{Z}$-basis of $\mathrm{End}(E)$ is given by

$$\{1, \quad \mathbf{i}, \quad (\mathbf{i}+\mathbf{j})/2, \quad (1+\mathbf{ij})/2\}\,.$$

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Assume $p \equiv 2 \pmod 3$.

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As far as we know, these are *hard* problems (even quantumly).

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$\rightsquigarrow$ **Approach #1**: Find “enough” isogeny cycles.

(Then find relations between them and construct an embedding into $B_{p,\infty}$.)

From concrete to abstract endomorphisms

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This defines an isometry $\operatorname{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} B_{p,\infty}$ w.r.t. $\langle \cdot, \cdot \rangle$.

Endomorphism rings via isogeny finding

From any isogeny $\varphi\colon E_0 \to E$, we obtain (abstractly) an embedding of the endomorphism ring

$$\begin{aligned} \operatorname{End}(E) &\hookrightarrow \overbrace{\operatorname{End}(E_0) \otimes_{\mathbb{Z}} \mathbb{Q}}^{=:B_{p,\infty}}\,; \\ \alpha &\mapsto \widehat{\varphi}\alpha\varphi/\deg(\varphi)\,. \end{aligned}$$

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Under this embedding, $\mathrm{End}(E) = \mathcal{O}_R(I) = \{\alpha \in B_{p,\infty} : I\alpha \subseteq I\}$, where $I := \mathrm{Hom}(E, E_0)\varphi$ is the ideal of E_0 associated to φ.

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$\rightsquigarrow$ **Approach #2**: Choose "special" E_0, find an isogeny $E_0 \to E$.
(Then recover the associated ideal and the codomain endomorphism ring.)

The Delfs–Galbraith algorithm

...is probably(?) the best known algorithm for solving the supersingular isogeny problem:

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All three steps can be done in time $\widetilde{O}(\sqrt{p})$...

- ...in a trivially parallelizable manner;

The Delfs–Galbraith algorithm

...is probably(?) the best known algorithm for solving the supersingular isogeny problem:

Given $E_0, E_1/\mathbb{F}_{p^2}$ supersingular, find an isogeny $E_0 \to E_1$.

1. Find any isogeny $\varphi_0 \colon E_0 \to E_0'$ where $E_0'/\mathbb{F}_p$.
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All three steps can be done in time $\widetilde{O}(\sqrt{p})$...

- ...in a trivially parallelizable manner;
- ...with negligible memory requirements.

Dividing endomorphisms

!! The representation of $\alpha \in \mathrm{End}(E)$ we obtain is fractional:

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$\Longrightarrow$ This is a (practically) very inconvenient representation.

From curves to quaternions

Plenty of algorithms to compute $E \mapsto \mathrm{End}(E)$ in time $\widetilde{O}(\sqrt{p})$.

It is not immediately obvious which one works best in practice.

From quaternions to curves

$$\mathcal{O} \mapsto E$$

(The "easy" direction.)

The main theorem

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Corollary (Deuring). Isomorphism classes of supersingular elliptic curves are in bijection with the (left) class set $\mathrm{Cls}_L(\mathcal{O}_0)$.

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<u>Important consequence:</u> The isogeny $\varphi_I \colon E_0 \to E$ defined by a left $\mathcal{O}_0$-ideal I has kernel $\bigcap_{\alpha \in I} \ker \alpha \le E_0$.

History and algorithms

- **1941**: Deuring proves the correspondence.

Wenn aber **R** eine vorgegebene Maximalordnung in $Q_{\infty,p}$ ist, in der der Primteiler von p Hauptideal ist, so gibt es genau eine Invariante j, zu der dieser Multiplikatorenring gehört, sie ist absolut rational. Ist der Primteiler von p kein Hauptideal, so gibt es zwei konjugierte Invarianten vom Absolutgrad 2 zu diesem Multiplikatorenring. Die Anzahl der j, zu denen eine Maximalordnung von $Q_{\infty,p}$ als Multiplikatorenring gehört, ist gleich der Klassenzahl von $Q_{\infty,p}$.

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From quaternions to curves

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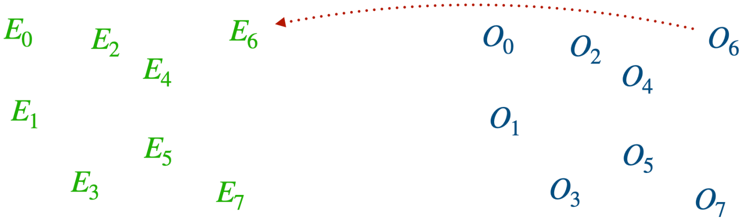

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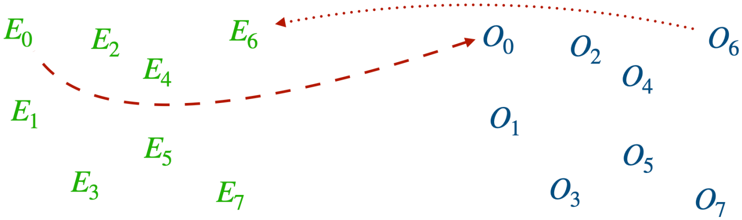

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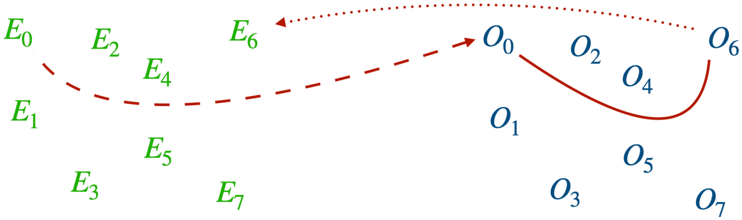

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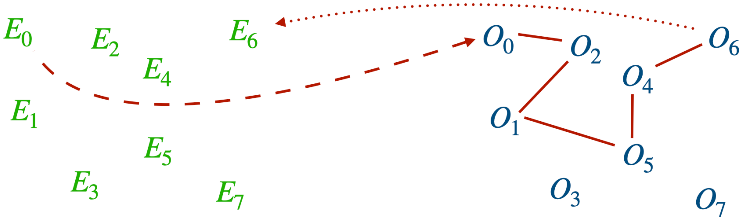

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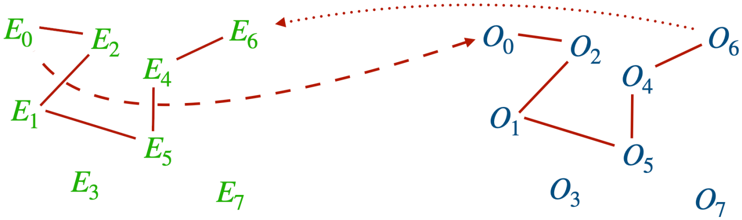

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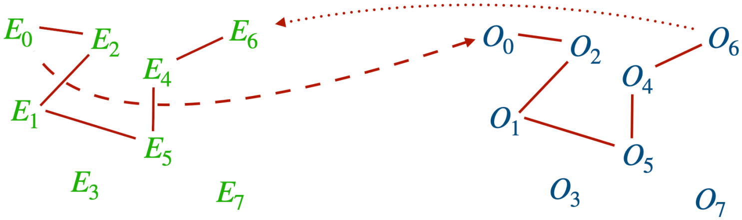

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I will talk about these *in reverse order*.

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Crucial observation: Complexity depends on factorization of N.

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2. That's all, but typically the norm of $\mathcal{O}\mathcal{O}'$ is horrible.
   (Also, it's integral only in trivial cases $\rightsquigarrow$ scale by denominator in $\mathbb{Z}$.)

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$\rightsquigarrow$ Do it twice with coprime degrees to evaluate on any point.

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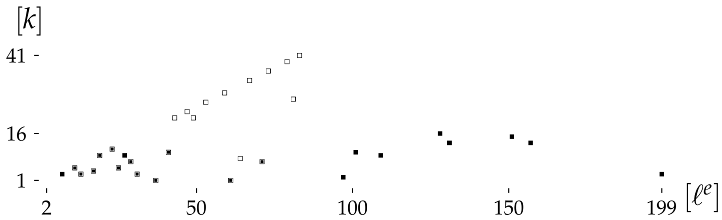
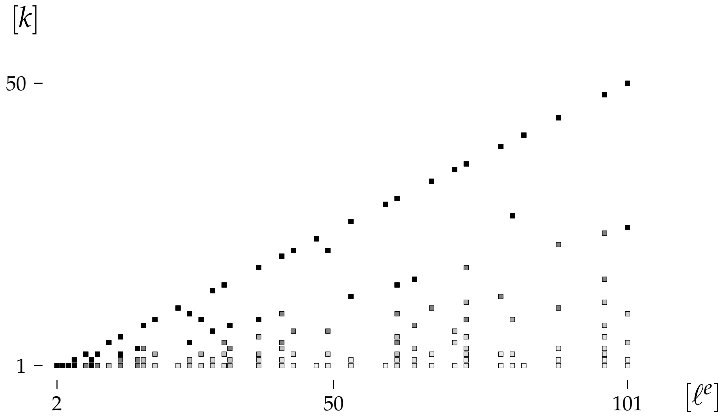


Heatmap



Average extension k required to access ℓ^e-torsion.

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Algorithm 5: `PushSubgroup`(E, f, φ)

Input: Elliptic curve $E/\mathbb{F}_q$, minimal polynomial $f \in \mathbb{F}_q[X]$ of a subgroup $G \leq E$, isogeny $\varphi\colon E \to E'$ defined over $\mathbb{F}_q$.

Output: Minimal polynomial $f^\varphi \in \mathbb{F}_q[X]$ of the subgroup $\varphi(G) \leq E'$.

Write the x-coordinate map of φ as a fraction g_1/g_2 of polynomials $g_1, g_2 \in \mathbb{F}_q[X]$.
Let $g_{\ker} \leftarrow \gcd(g_2, f)$ and $f_1 \leftarrow f/g_{\ker}$.
Compute $g_1 \cdot g_2^{-1} \bmod f_1 \in \mathbb{F}_q[X]$ and reinterpret it as a quotient-ring element $\alpha \in \mathbb{F}_q[X]/f_1$.
Find the minimal polynomial $f^\varphi \in \mathbb{F}_q[X]$ of α over $\mathbb{F}_q$ using Shoup's algorithm.
Return f^φ.

Cool trick #2: Isogenies from minimal polynomials

Another algorithmic trick for kernel *points* lying in extensions:

- Replace (big) kernel polynomials by possibly much smaller minimal polynomials of isogenies.
  They are irreducible divisors of the kernel polynomial.
- Shoup's algorithm gives a fast method to push minimal polynomials through isogenies. $\rightsquigarrow$ Evaluating isogeny chains.

Algorithm 5: $\texttt{PushSubgroup}(E, f, \varphi)$

Input: Elliptic curve $E/\mathbb{F}_q$, minimal polynomial $f \in \mathbb{F}_q[X]$ of a subgroup $G \leq E$, isogeny $\varphi\colon E \to E'$ defined over $\mathbb{F}_q$.

Output: Minimal polynomial $f^\varphi \in \mathbb{F}_q[X]$ of the subgroup $\varphi(G) \leq E'$.

Write the x-coordinate map of φ as a fraction g_1/g_2 of polynomials $g_1, g_2 \in \mathbb{F}_q[X]$.
Let $g_{\text{ker}} \leftarrow \gcd(g_2, f)$ and $f_1 \leftarrow f/g_{\text{ker}}$.
Compute $g_1 \cdot g_2^{-1} \bmod f_1 \in \mathbb{F}_q[X]$ and reinterpret it as a quotient-ring element $\alpha \in \mathbb{F}_q[X]/f_1$.
Find the minimal polynomial $f^\varphi \in \mathbb{F}_q[X]$ of α over $\mathbb{F}_q$ using Shoup's algorithm.
Return f^φ.

Complexity: $O(k^2) + \widetilde{O}(n)$. Naïvely $O(nk(\log k)^{O(1)})$.

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  Often easy to explicitly write down; tricky in general $\left(p \equiv 1 \pmod{12}\right)$.

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  Find q such that $\mathbf{i}^2 = -q$, $\mathbf{j}^2 = -p$ defines $B_{p,\infty}$, find a root $j \in \mathbb{F}_p$ of the Hilbert class polynomial H_{-q}, construct a curve with this j-invariant.

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  Algorithm to compute a *normalized* degree-q isogeny in time $\widetilde{O}(q)$.
  Composing the desired endomorphism $\vartheta\colon E \to E$ with the isomorphism $\tau\colon (x,y) \mapsto (-qx, \sqrt{-q}^3 y)$ makes it normalized.

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- Ingredient #3: Ibukiyama's theorem.
  Explicit basis for a maximal order of $B_{p,\infty}$ with an endomorphism $\sqrt{-q}$.
  In fact, there are only very few maximal orders containing $\sqrt{-q}$.

Open-source code

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    + 7076*k, -1722016565/2 + 1401001825/2*i + 551/2*j
    + 16579/2*k, -2147483647 - 9708*j + 12777*k, -2147483647
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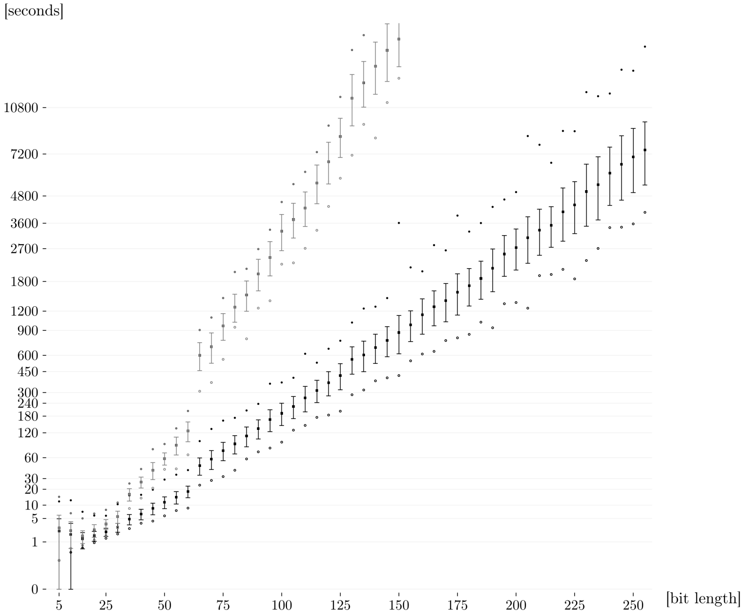
sage: E1, phi, _ = constructive_deuring(I, E0, iota)
sage: phi
Composite morphism of degree 14763897348161206530374369280
                = 2^29*3^3*5*7^2*11*13*17*31*41*43^2*61*79*151:
  From: Elliptic Curve defined by y^2 = x^3 + x over
            Finite Field in i of size 2147483647^2
  To:   Elliptic Curve defined by y^2 = x^3 + (1474953432*i
                +1816867654)*x + (581679615*i+260136654)
            over Finite Field in i of size 2147483647^2
```

Timings (SageMath, single core)



Timings (SageMath, single core)

We've been informed of one run for a ≈ 500-bit characteristic that took only about 7 hours.

$\rightsquigarrow$ Definitely practical for parameter setup, cryptanalysis, etc.!

Questions?

(Also feel free to email me: `lorenz@yx7.cc`)