Rational isogenies from irrational endomorphisms

Wouter Castryck[1] Lorenz Panny[2] Frederik Vercauteren[1]

[1]imec-COSIC, ESAT, KU Leuven [2]Academia Sinica, Taipei, Taiwan

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See also very much related parallel work by Boneh and Love [arXiv:1910.03180].

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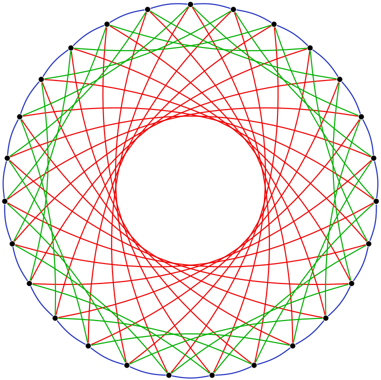
$\Longrightarrow$ Bottom line: Relatively fast non-interactive key exchange.
Think Diffie–Hellman, but post-quantum! (and slower...)

Isogeny graphs

Visualizing the action of $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$:



Each node is an elliptic curve over $\mathbb{F}_p$, up to $\cong_{\mathbb{F}_p}$.

Each edge is the action of $\mathfrak{l}_1$, $\mathfrak{l}_2$, or $\mathfrak{l}_3$, or their inverses.

Notation for this talk

- The prime p is 'large', certainly > 3.
- Curves are elliptic, supersingular, and defined over $\mathbb{F}_{p^2}$.
- E^t: the *quadratic twist* of E.
- $\mathrm{End}(E)$: *full* endomorphism ring (over $\bar{\mathbb{F}}_p$).
- $\mathrm{End}_p(E)$: *rational* endomorphism ring (over $\mathbb{F}_p$).
- E_0: a starting curve with known endomorphism ring.
  For instance: $p \equiv 3 \bmod 4$ and $E_0 : y^2 = x^3 + x$.
  Endomorphism ring: $\mathrm{End}(E_0) = \left\langle 1, \iota, \frac{\iota+\pi}{2}, \frac{1+\iota\pi}{2} \right\rangle$
  where $\iota : (x, y) \mapsto (-x, \sqrt{-1} \cdot y)$ and $\pi : (x, y) \mapsto (x^p, y^p)$.
- $\mathcal{O}$: the order $\mathbb{Z}[\sqrt{-p}]$ or $\mathbb{Z}[(1+\sqrt{-p})/2]$ in $\mathbb{Q}(\sqrt{-p})$.
- $\mathfrak{l}$: a fixed prime ideal of $\mathcal{O}$ lying above ℓ.

A starting point...

Suppose a curve $E = [\mathfrak{a}]E_0$ has an irrational endomorphism $\tau \in \mathrm{End}(E) \backslash \mathrm{End}_p(E)$, say of prime degree ℓ.

Q: Where in the isogeny graph is it?

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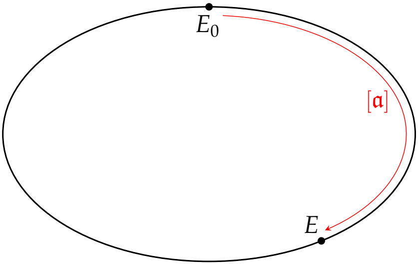

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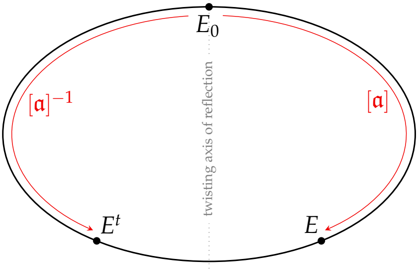

Fact: *If* $p \equiv 3 \bmod 4$ and $E_0 : y^2 = x^3 + x$, then "given $[\mathfrak{a}]E_0$ we can compute $[\mathfrak{a}]^{-1}E_0$ by mere quadratic twisting". [CSIDH paper]

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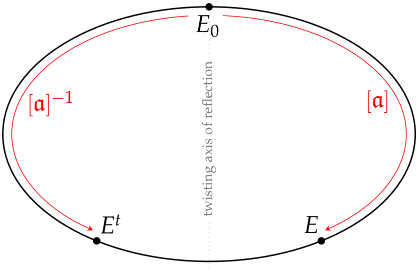

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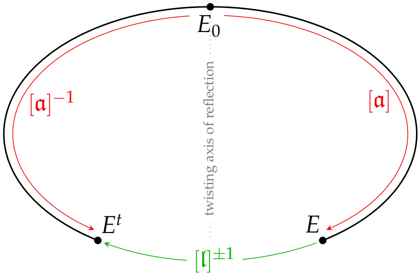

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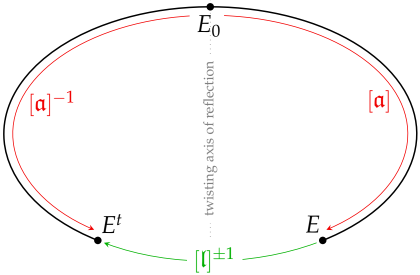

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Previous slide:

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Definition. Let E be defined over $\mathbb{F}_p$.
Then $\alpha \in \mathrm{End}(E)$ is a *twisting endomorphism* of E if $\alpha\pi = -\pi\alpha$.

To-do list

Let $E = [\mathfrak{a}]E_0$. We've seen:

If $p \equiv 3 \bmod 4$ **and** $E_0\colon y^2 = x^3 + x$ **and** $\tau \in \mathrm{End}(E) \backslash \mathrm{End}_p(E)$ with $\deg \tau = \ell$ prime **and** $\tau\pi = -\pi\tau$, **then** $[\mathfrak{a}]^2 = [\mathfrak{l}]^{\pm 1}$.

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Note: *If* the class number $h(\mathcal{O}) = |\mathrm{cl}(\mathcal{O})|$ is known and odd, then

$$\sqrt{[\mathfrak{s}]} = [\mathfrak{s}]^{(h(\mathcal{O})+1)/2}.$$

Gauß' algorithm does not require computing $h(\mathcal{O})$.

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For the potential divisors of Δ, we get:

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$$\implies \mathrm{cl}(\mathcal{O})[2] \cong \begin{cases} \{\mathrm{id}\} & \text{when } p \equiv 3 \pmod 4; \\ \mathbb{Z}/2 & \text{when } p \equiv 1 \pmod 4. \end{cases}$$

Bottom line: Elements $[\mathfrak{s}] \in \mathrm{cl}(\mathcal{O})^2$ have either one or two square roots, depending on $p \bmod 4$.

To-do list

- How to compute square roots in $\mathrm{cl}(\mathcal{O})$? ✓
  Gauß found a polynomial-time algorithm.
- How much ambiguity is in the 2-torsion? ✓
  At most two square roots; $\mathrm{cl}(\mathcal{O})[2] \leq \mathbb{Z}/2$.
- When are endomorphisms *twisting*?
- Can we deal with starting curves $E_0 \neq E_0^t$?
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Moreover, given any irrational endomorphism, it is typically easy to find a twisting endomorphism.

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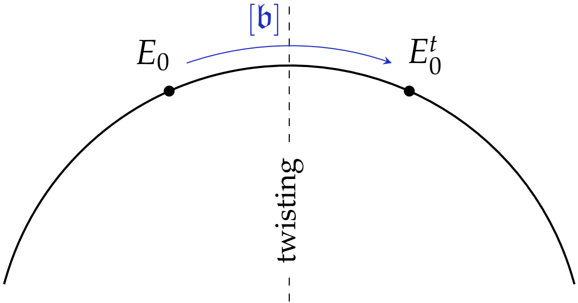

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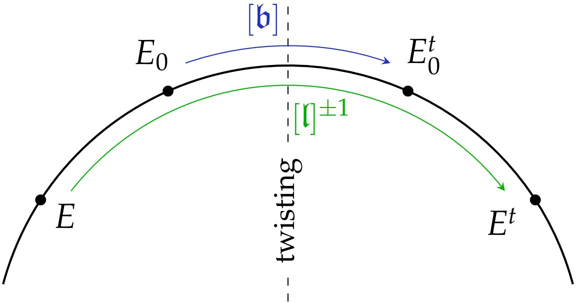

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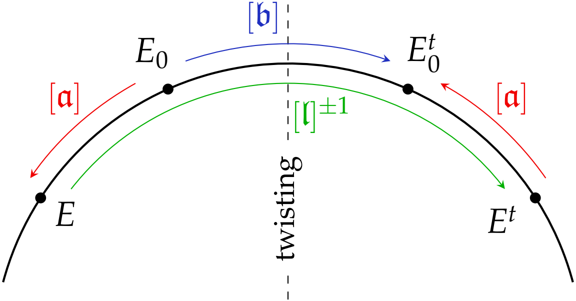

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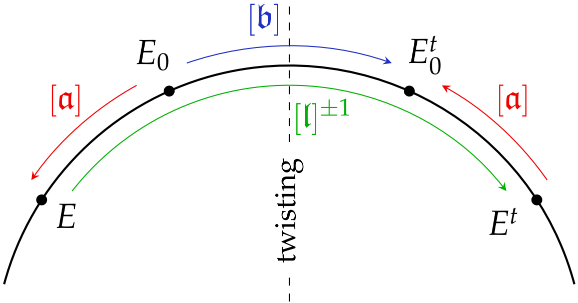
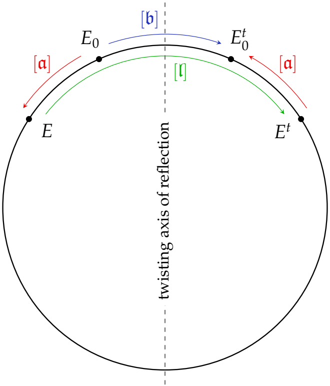


$$[\mathfrak{l}]^{\pm 1} = [\mathfrak{b}][\mathfrak{a}]^{-2} \implies [\mathfrak{a}]^2 = [\mathfrak{b}][\mathfrak{l}]^{\mp 1}$$

To-do list

- How to compute square roots in $\mathrm{cl}(\mathcal{O})$? ✓
  Gauß found a polynomial-time algorithm.
- How much ambiguity is in the 2-torsion? ✓
  At most two square roots; $\mathrm{cl}(\mathcal{O})[2] \leq \mathbb{Z}/2$.
- When are endomorphisms *twisting*? ✓
  Sufficient: reduced CM endomorphisms with $\deg \leq (p+1)/4$.
- Can we deal with starting curves $E_0 \neq E_0^t$? ✓
  Yes; the same idea works modulo technicalities.
- Can we generalize to primes $p \not\equiv 3 \bmod 4$?

The case $p \equiv 1 \bmod 4$



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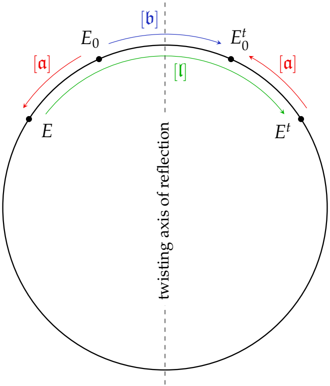

<u>Long story short</u>: Everything works the same, but the element $\mathfrak{t} := [(2, \pi+1)]$ of order 2 introduces an additional symmetry.

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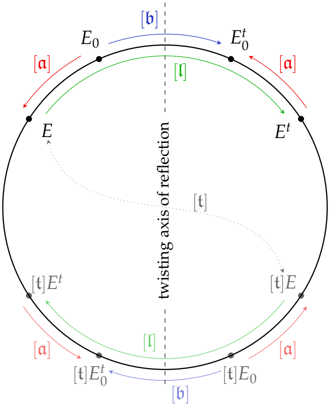

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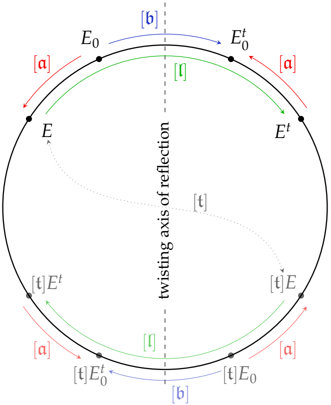

Long story short: Everything works the same, but the element $\mathfrak{t} := [(2, \pi+1)]$ of order 2 introduces an additional symmetry.

$\rightsquigarrow$ Two candidates for $[\mathfrak{a}]$. Find $[\mathfrak{a}]$ by brute-force testing or use ePrint 2020/151, which breaks DDH for the case $p \equiv 1 \bmod 4$.

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Our ‘locating CM curves’ theorem

Let $p \equiv 3 \bmod 4$ and $\ell < (p+1)/4$ be primes with $\left(\frac{-p}{\ell}\right) = 1$.

We show:

- How many curves $/\mathbb{F}_p$ are reductions of curves $/\bar{\mathbb{Q}}$ with CM by orders $\mathcal{R} \subseteq \mathbb{Q}(\sqrt{-\ell})$ containing $\mathbb{Z}[\sqrt{-\ell}]$.
- Which combinations of $(\mathrm{End}_p, \mathcal{R})$ are possible.
- Where in the isogeny graph all these curves are located: We give connecting ideals to the curve $E_0 \colon y^2 = x^3 \pm x$.

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Remark: Similar results are possible for $p \equiv 1 \bmod 4$.

An example

In the CSIDH-512 parameter set, $p \equiv 11 \bmod 12$.
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This ideal class corresponds to (e.g.) the private key:

$$\begin{array}{l}(5, -7, -1, 1, -4, -5, -8, 4, -1, 5, 1, 0, -2, -4, -2, 2, -9, 4, 2, \\ 5, 1, 1, 1, 5, -4, 2, 6, 5, -1, 0, 0, -4, -1, -3, -1, -4, 1, 7, \\ 1, 4, 1, 4, -7, 0, -3, -1, 0, 1, 2, 3, 1, 2, -4, -5, 9, -1, 4, \\ 0, 5, 1, 0, 1, 1, 3, 0, 2, 2, 2, -1, 2, 1, -1, 11, 3).\end{array}$$

[relies on data from ePrint 2019/498]

One last thing: $\mathbb{F}_p$-ifying the KLPT algorithm

Let E be a supersingular elliptic curve.

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  - But this might be optimal: we show that doing better implies faster discrete logarithms in $\mathrm{cl}(\mathbb{Q}(\sqrt{-p}))$.

Thanks!

Further reading for any newcomers who may have now acquired an interest in the actual isogeny session (later today):

- `https://arxiv.org/abs/1711.04062`
- `https://yx7.cc/docs/phd/thesis.pdf` (§2)