CSIDH:
An Efficient Post-Quantum Commutative Group Action

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- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim 85\,\mathrm{ms}$ for a full key exchange
- Flexible: compatible with 0-RTT protocols such as QUIC; recent preprint uses CSIDH for 'SeaSign' signatures

Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group G via the map

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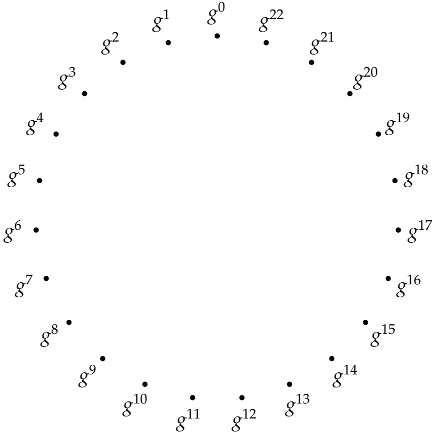
$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Square-and-multiply

Suppose $G \cong \mathbb{Z}/23$ and that Alice computes g^{13}.



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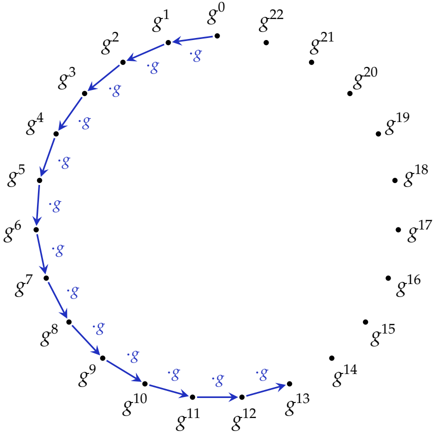

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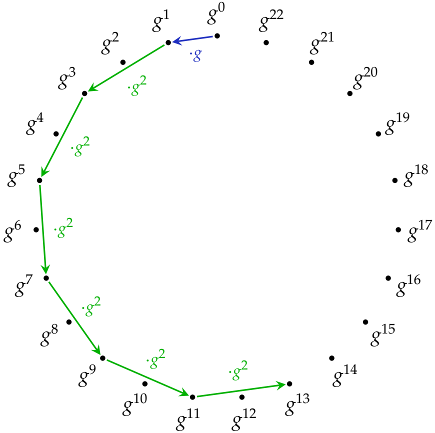

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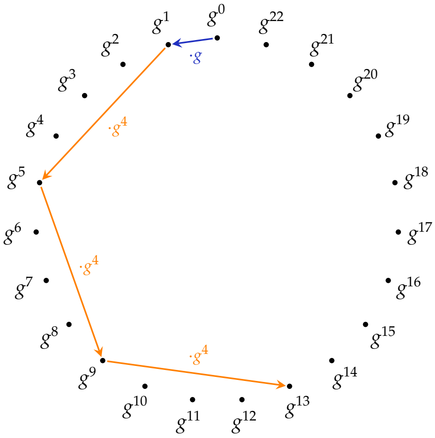

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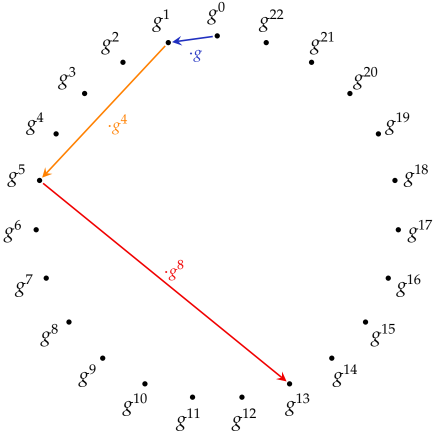
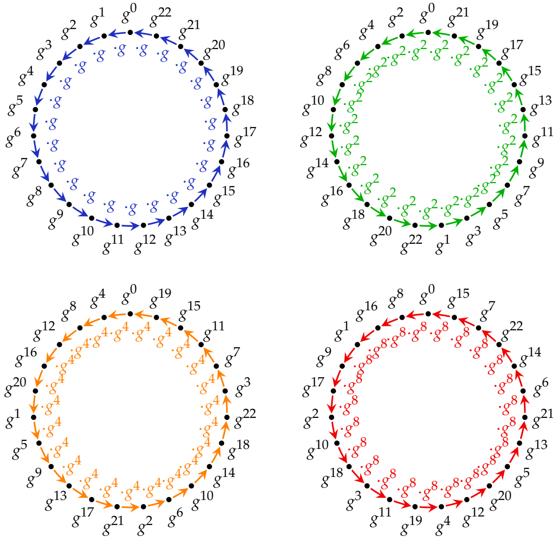


Cycles!



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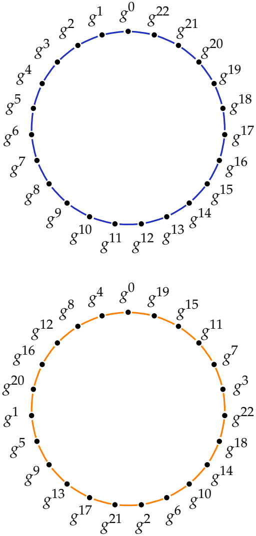

g^{0}
g^{22}
g^{21}
g^{20}
g^{19}
g^{18}
g^{17}
g^{16}
g^{15}
g^{14}
g^{13}
g^{12}
g^{11}
g^{10}
g^{9}
g^{8}
g^{7}
g^{6}
g^{5}
g^{4}
g^{3}
g^{2}
g^{1}
g^{0}
g^{21}
g^{19}
g^{17}
g^{15}
g^{13}
g^{11}
g^{9}
g^{7}
g^{5}
g^{3}
g^{1}
g^{22}
g^{20}
g^{18}
g^{16}
g^{14}
g^{12}
g^{10}
g^{8}
g^{6}
g^{4}
g^{2}
g^{0}
g^{19}
g^{15}
g^{11}
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g^{6}
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g^{21}
g^{17}
g^{13}
g^{9}
g^{5}
g^{1}
g^{20}
g^{16}
g^{12}
g^{8}
g^{4}
g^{0}
g^{15}
g^{7}
g^{22}
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g^{21}
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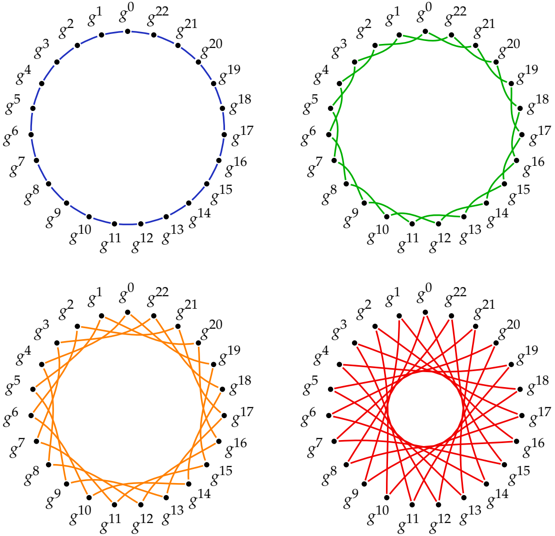
g^0 g^1 g^2 g^3 g^4 g^5 g^6 g^7 g^8 g^9 g^{10} g^{11} g^{12} g^{13} g^{14} g^{15} g^{16} g^{17} g^{18} g^{19} g^{20} g^{21} g^{22}
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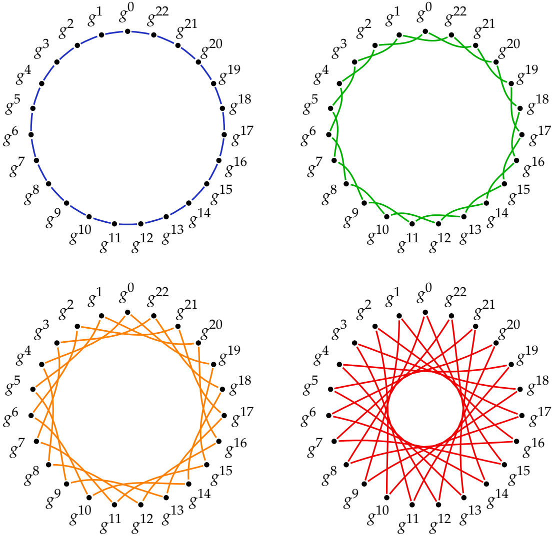
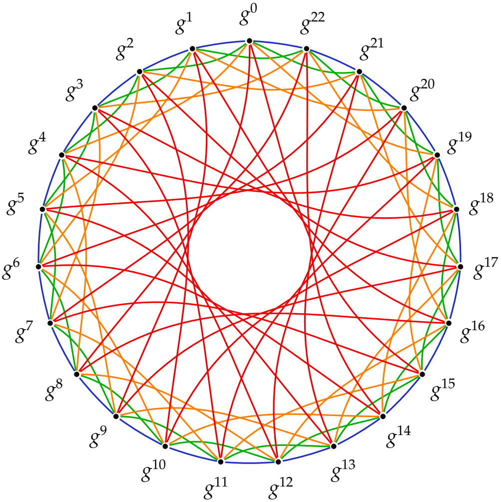


Cycles are compatible: [right, then left] = [left, then right], etc.

Union of cycles: rapid mixing



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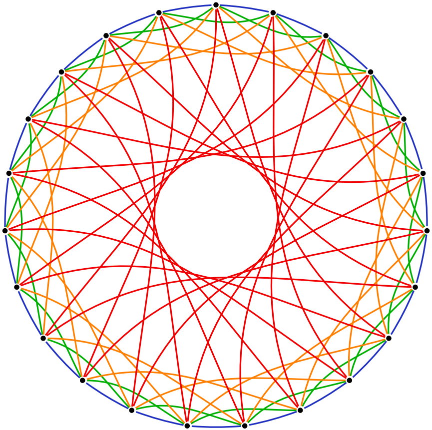
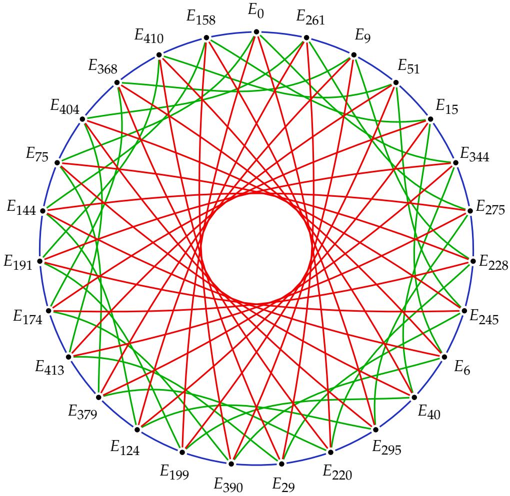
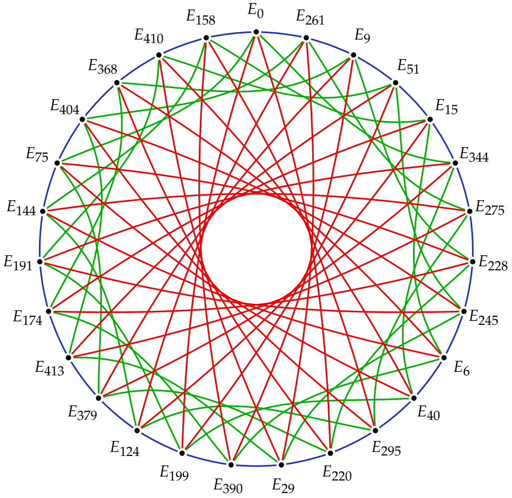


CSIDH: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



Graphs of elliptic curves



Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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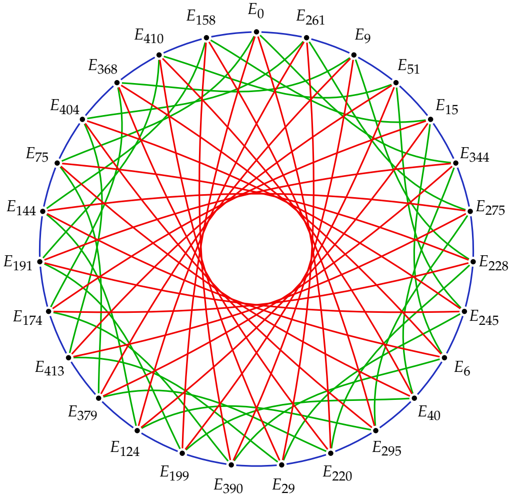

Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies (certain kinds of maps).

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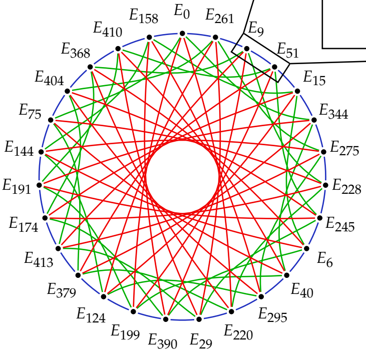
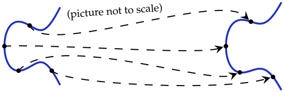


A 3-isogeny



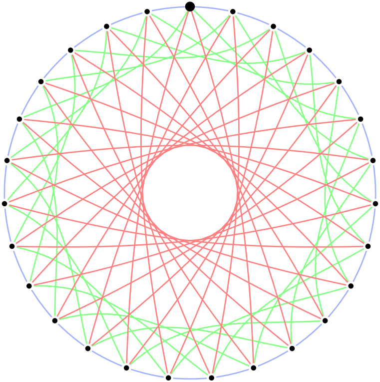
$$E_{51}\colon y^2 = x^3 + 51x^2 + x \longrightarrow E_9\colon y^2 = x^3 + 9x^2 + x$$

$$(x, y) \longmapsto \left(\frac{97x^3 - 183x^2 + x}{x^2 - 183x + 97}, \; y \cdot \frac{133x^3 + 154x^2 - 5x + 97}{-x^3 + 65x^2 + 128x - 133} \right)$$

Diffie-Hellman on 'nice' graphs

Alice

$[+, -, +, -]$



Bob

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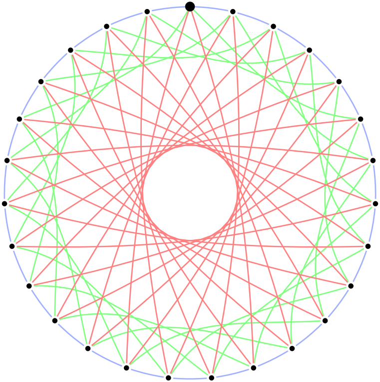

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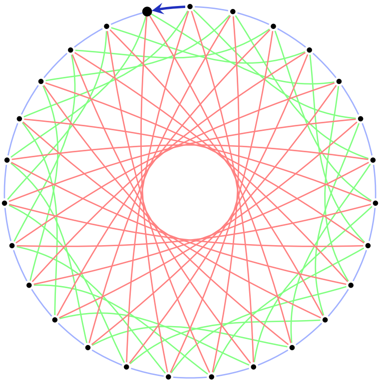

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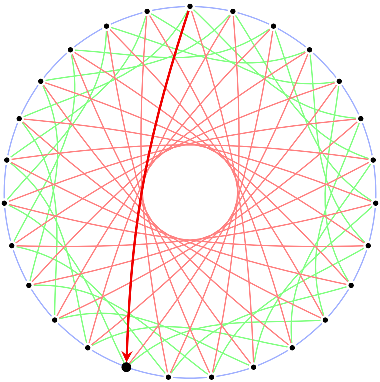

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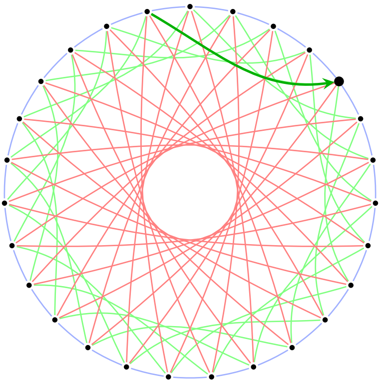

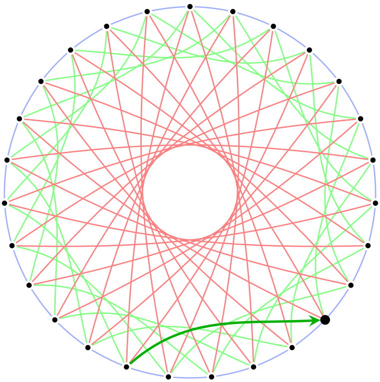

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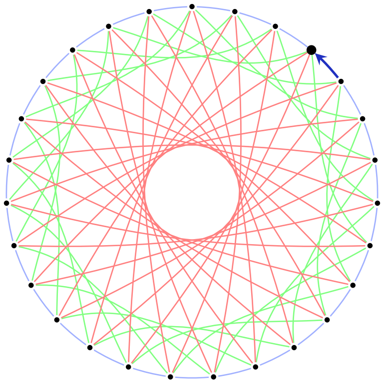

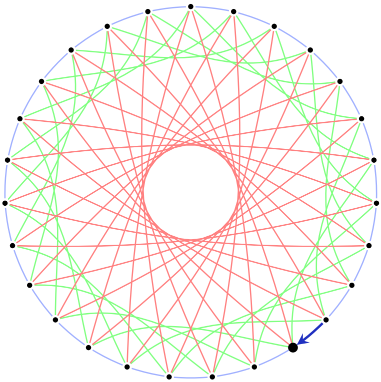

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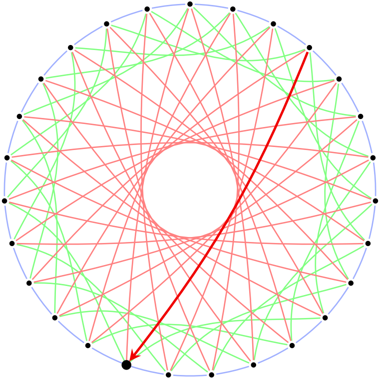

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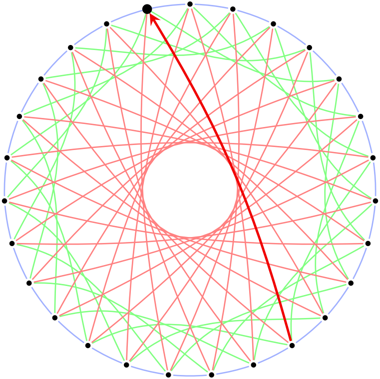

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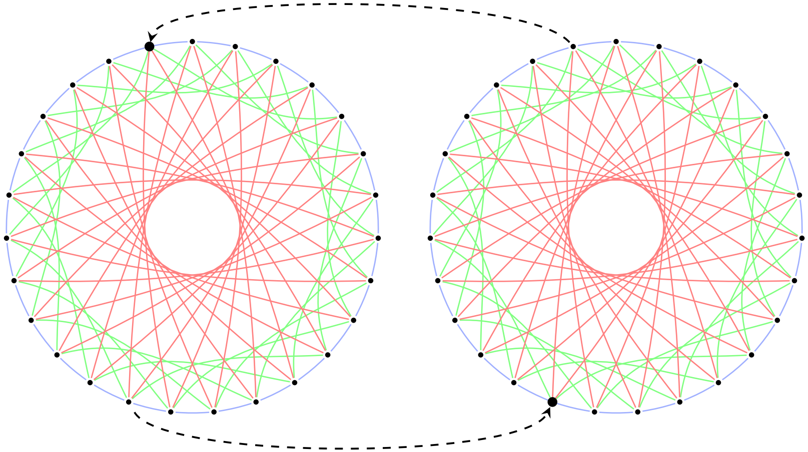

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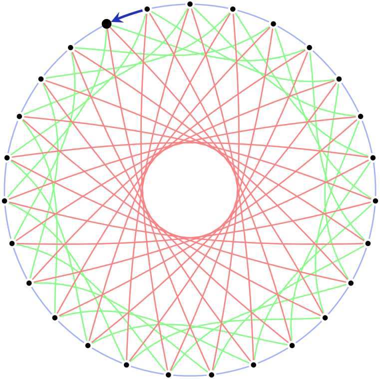

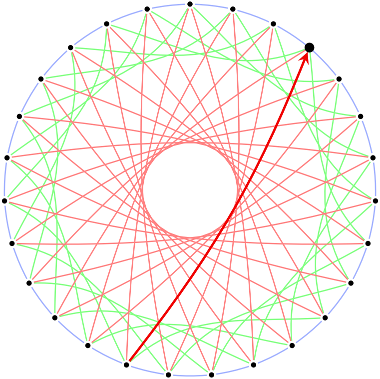

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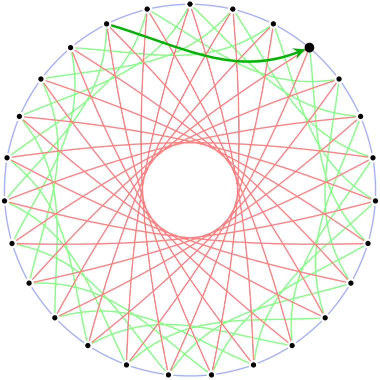

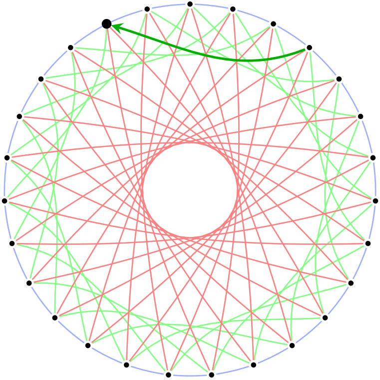

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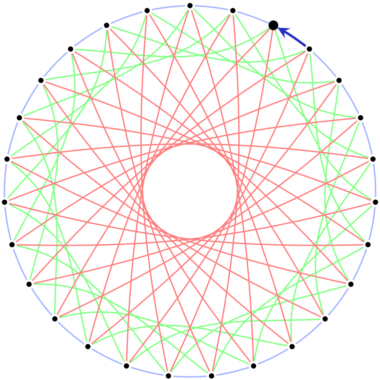

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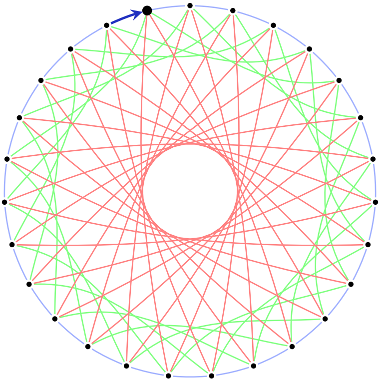

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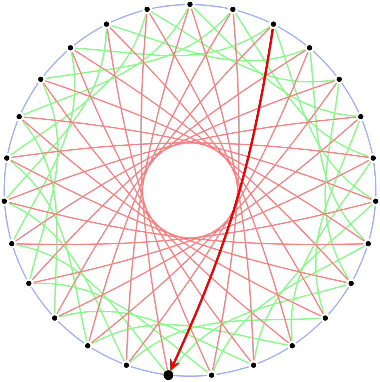

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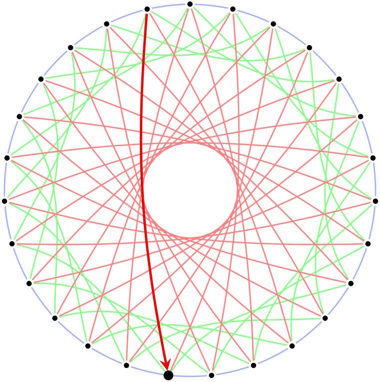

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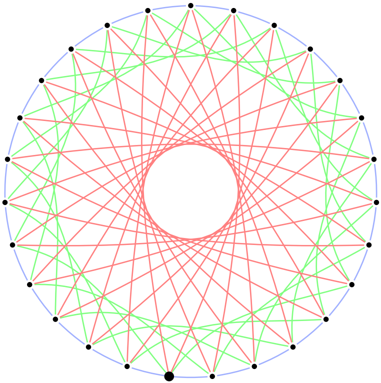

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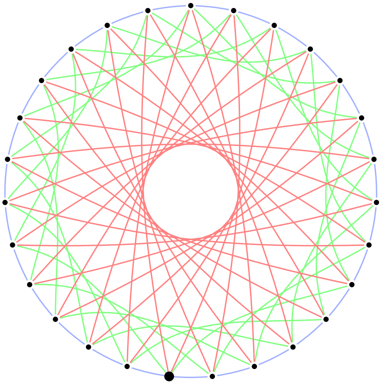

A walkable graph

Important properties for such a walk:

IP1 ▸ The graph is a composition of compatible cycles.

IP2 ▸ We can compute neighbours in given directions.

IP1: A composition of cycles

- The graph used in CSIDH is constructed as a composition of graphs G_ℓ of 'ℓ-isogenies'.

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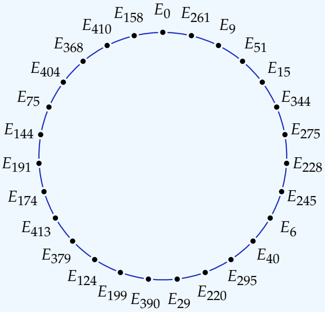
- In our example, these are

G_3:



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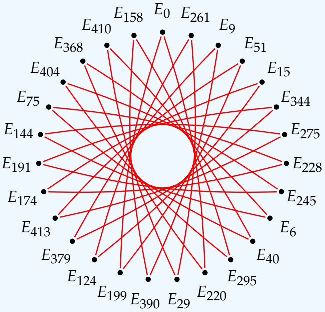

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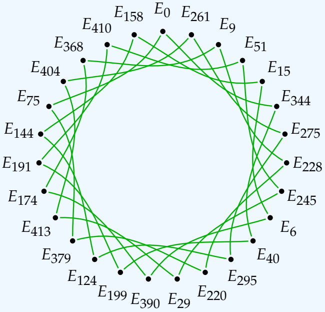

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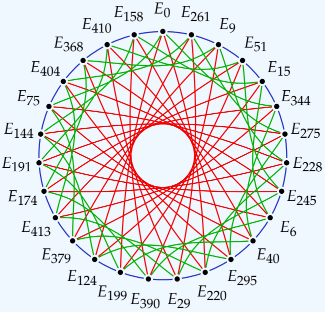
$G_3 \cup G_5 \cup G_7$:



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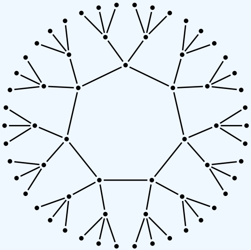
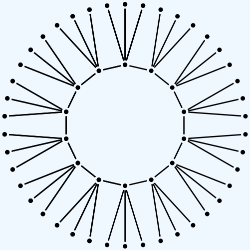
G_3:  G_5: 

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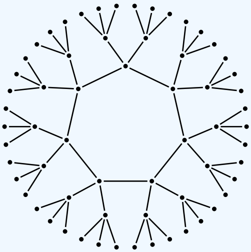

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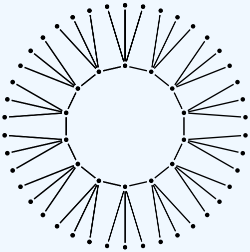
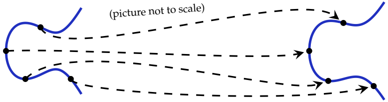


- We want to make sure G_ℓ is just a cycle.

IP2: Compute neighbours in given directions

The edges of G_ℓ are ℓ-isogenies.



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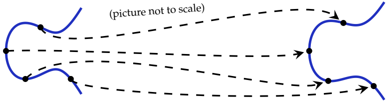

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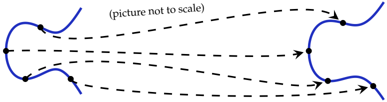

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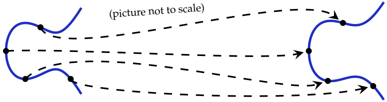

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- Generally needs big extension fields...

Point counting

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It seems difficult to find a curve with a given number of points (and such that the graph is big). [De Feo–Kieffer–Smith]

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- All G_{ℓ_i} are compatible.
- Computations need only $\mathbb{F}_p$-arithmetic.

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Side effect of magic:

- Every node of G_{ℓ_i} can be written as

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$\Rightarrow$ Can compress every node to a single value $A \in \mathbb{F}_p$.

$\Rightarrow$ Tiny keys!

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[1] This algorithm has a small chance of false positives, but we actually use a variant that *proves* that E_A has $p + 1$ points.

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Does any A work?

No.

- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p+1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p+1]P = \infty$.[1]

[1] This algorithm has a small chance of false positives, but we actually use a variant that *proves* that E_A has $p+1$ points.

Security

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  - Concrete estimates are to be done.
  - (Recent preprint [BS] ignores much of the cost!)

Parameters

CSIDH-log p	target NIST level	public key size	private key size	time (full exchange)	cycles (full exchange)	stack memory	classical security	quantum security claimed by [BS] (take *cum grano salis*)
CSIDH-512	1	64 b	32 b	85 ms	212e6	4368 b	128	71
CSIDH-1024	3	128 b	64 b				256	88
CSIDH-1792	5	224 b	112 b				448	104

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- [Your paper here!]


Thank you!

References

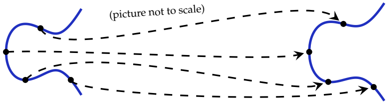
Mentioned in this talk:

- Castryck, Lange, Martindale, Panny, Renes:
  CSIDH: An Efficient Post-Quantum Commutative Group Action
  `https://ia.cr/2018/383` (to appear at ASIACRYPT 2018)
- De Feo, Kieffer, Smith:
  Towards practical key exchange from ordinary isogeny graphs
  `https://ia.cr/2018/485` (to appear at ASIACRYPT 2018)
- De Feo, Galbraith:
  SeaSign: Compact isogeny signatures from class group actions
  `https://ia.cr/2018/824`
- [BS] Bonnetain, Schrottenloher:
  Quantum Security Analysis of CSIDH and Ordinary Isogeny-based Schemes
  `https://ia.cr/2018/537`

Other related work:

- Delfs, Galbraith:
  Computing isogenies between supersingular elliptic curves over $\mathbb{F}_p$
  `https://arxiv.org/abs/1310.7789`
- Childs, Jao, Soukharev:
  Constructing elliptic curve isogenies in quantum subexponential time
  `https://arxiv.org/abs/1012.4019`
- Meyer, Reith:
  A faster way to the CSIDH
  `https://ia.cr/2018/782`
  (to appear at Indocrypt 2018)
- Jao, LeGrow, Leonardi, Ruiz-Lopez:
  A polynomial quantum space attack on CRS and CSIDH
  (MathCrypt 2018)
- Biasse, Iezzi, Jacobson:
  A note on the security of CSIDH
  `https://arxiv.org/abs/1806.03656`
  (to appear at Indocrypt 2018)

Where's the group?



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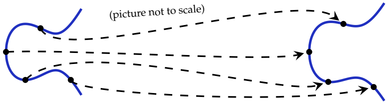

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- $E_9 = E_{51}/\mathfrak{a}$ where $\mathfrak{a}$ is the ideal $(3, \pi - 1)$ of $\mathrm{End}_{\mathbb{F}_p}(E_{51})$.
- For our choices of A, $\mathrm{End}_{\mathbb{F}_p}(E_A) \cong \mathbb{Z}[\sqrt{-p}]$.
- The group action is

$$\begin{aligned} \mathrm{cl}(\mathbb{Z}[\sqrt{-p}]) \times \{E_A\} &\longrightarrow \{E_A\} \\ ([\mathfrak{a}], E) &\longmapsto E/\mathfrak{a} \end{aligned}$$

(modulo details).