CSIDH:
An Efficient Post-Quantum Commutative Group Action

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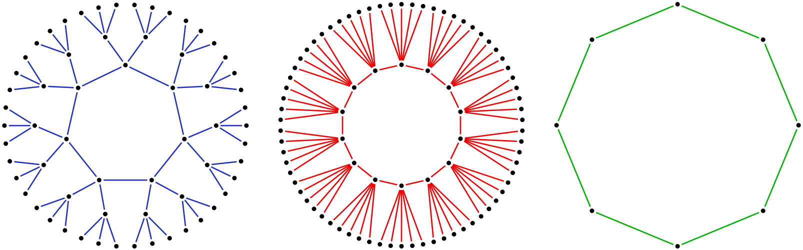
- Clean mathematical structure: a true group action. (No noise, no auxiliary points, no compromises.)
- By the way: not 'better' or 'worse' than SIDH. It's simply different and likely to be useful for different applications.

Ordinary isogeny graphs

Nodes: Ordinary elliptic curves defined over k up to $\cong_k$.
Edges: 3-, 5-, and 7-isogenies defined over k up to $\cong_k$.

Components look something like this:



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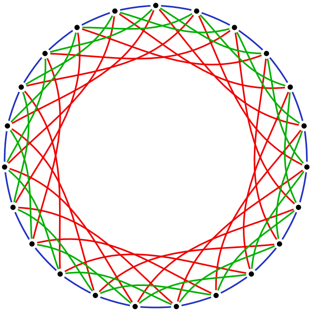

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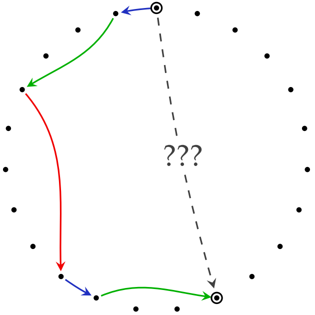
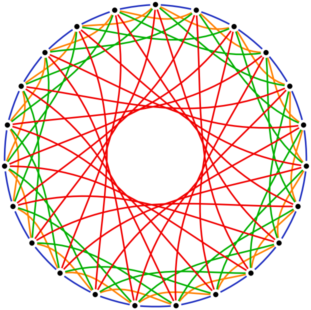


Easy: Compute a random path, output the final node.
Hard problem: Find a path between two given nodes.

Alice vs. Eve



Intuition: Combining edges from different cycles allows taking shortcuts to remote parts of the graph!

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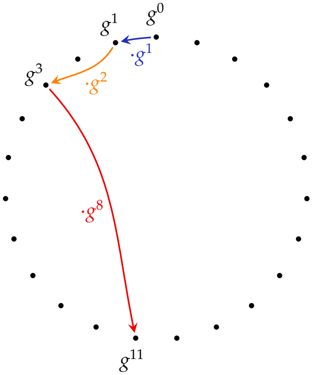

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cf. Square-&-Multiply: Alice gets an advantage over Eve.

Point counting

De Feo–Kieffer–Smith want
an ordinary curve $E/\mathbb{F}_q$ with many small primes $\ell \mid E(\mathbb{F}_q)$.

This seems difficult.

A crypto conference is never complete without ______.

Pictures: `https://github.com/CardsAgainstCryptography`

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Citing personal communication.

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Are there any security issues with using supersingular curves?
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Wouter, you are a genius! — me

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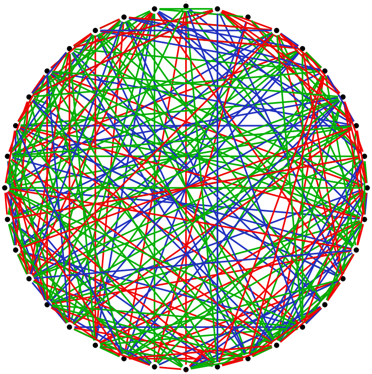

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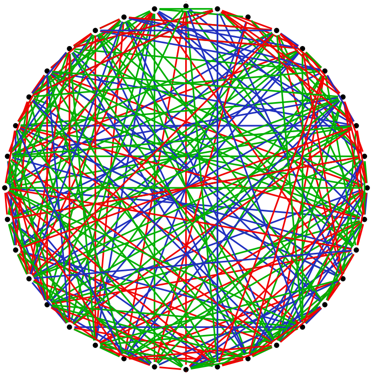

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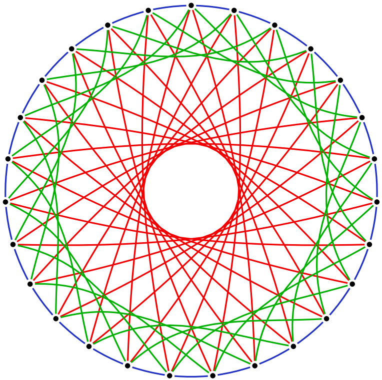

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Theorem/fact/definition. Let $p > 3$. An elliptic curve E over $\mathbb{F}_p$ is supersingular if and only if $\#E(\mathbb{F}_p) = p + 1$.

$\Longrightarrow$ We can simply craft a curve with a good number of points.

Reminder

The class group action is defined as follows:

- **Inputs**:
  An elliptic curve E with $\mathbb{F}_q$-endomorphism ring $\mathcal{O}$,
  an ideal $\mathfrak{a} \subseteq \mathcal{O}$ of prime norm ℓ.
- **Output**:
  The elliptic curve $[\mathfrak{a}]E$.

1. Compute the subgroup $E[\mathfrak{a}] = \bigcap_{\alpha \in \mathfrak{a}} \ker \alpha$ killed by $\mathfrak{a}$.
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Typically $E[\mathfrak{a}]$ is only defined over $\mathbb{F}_{q^m}$ for $m \approx \ell$.
$\Longrightarrow$ Complexity of computing with $E[\mathfrak{a}]$ is exponentiaℓ... ☹

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   - All curves in X have $\mathbb{F}_p$-endomorphism ring $\mathcal{O} = \mathbb{Z}[\sqrt{p}]$.
     Define the ideals $\mathfrak{l}_i = (\ell_i, \pi - 1)$ of $\mathcal{O}$.
   - Let $K = \{[\mathfrak{l}_1^{e_1} \cdots \mathfrak{l}_n^{e_1}] \mid (e_1, \ldots, e_n) \text{ is 'short'}\} \subseteq \mathrm{cl}(\mathcal{O})$.

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4.
- $\mathrm{cl}(\mathcal{O})$ acts on X and the action of K is very efficient!

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$\implies$ With x-only arithmetic everything can be done over $\mathbb{F}_p$.

Magic (public keys)

Theorem. For $p > 3$ and $p \equiv 3 \pmod 8$,
a supersingular elliptic curve over $\mathbb{F}_p$ can be written in the form

$$E_A\colon\ y^2 = x^3 + Ax^2 + x$$

if and only if the $\mathbb{F}_p$-rational endomorphism ring of E is $\mathbb{Z}[\sqrt{p}]$.
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  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p+1]P = \infty$.
  This algorithm has a negligible chance $8/\sqrt{p} + o(1)$ of false positives.
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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.

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Quantum:

- Hidden-shift algorithms: Subexponential complexity.
  - Literature contains mostly asymptotics.
  - Time-space trade-off: Fastest variants need huge memory.
  - [BS] ignores much of the cost. No need to panic!

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- Quantum: complicated. AFAWK it reaches NIST level 1.
  [BS] says $2^{32.5}$ queries; [BLMP] estimates $\approx 2^{81}$ quantum gates using millions of qubits.

Work in progress & future work

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- [Your paper here!]

Questions?

[CSIDH] https://ia.cr/2018/383
[BS] https://ia.cr/2018/537
[BLMP] https://ia.cr/2018/1059

SIDH vs. CSIDH

$$\text{CSIDH} = \text{SIDH?}$$

SIDH vs. CSIDH

$$\text{CSIDH} = \text{SIDH} + \text{C}$$

SIDH vs. CSIDH

Sizes and times are for (conjectured) NIST level 1.
SIDH parameters are more conservative.

	SIDH	**CSIDH**
Time per key exchange	$\approx 10\,\text{ms}$	$\approx 70\,\text{ms}$
Public keys	378 b	64 b
Public key compression	222 b ($\approx 15\,\text{ms}$)	n/a
Constant-time slowdown	≈ 1	≈ 6 (quick 'n' dirty)
In the NIST not-a-competition	yes	no
Maturity	7 years	7 months
Classical security	$p^{1/4}$	$p^{1/4}$
Quantum security	$p^{1/6}$	$L_p[1/2]$
$\rightsquigarrow$ Key size scaling	linear	quadratic
Chosen-ciphertext security (KEM)	generic transform	built-in
Non-interactive key exchange	slow	built-in
Signatures (now)	seconds	snail speed
Signatures (future?)	still seconds?	seconds

(slide mostly stolen from Chloe Martindale, who mostly stole it from Luca De Feo)